GENERALIZED CONCAVITY IN FUZZY OPTIMIZATION AND DECISION ANALYSIS

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Generalized Concavity in Fuzzy Optimization and Decision Analysis

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Contents

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List of Symbols

Symbol	*Description*
$\mathbf{R}^n$	n-dimensional (Euclidean) real vector space
$f : X \to Y$	mapping or function f that maps a set X into a set Y
$\mathrm{Ran}(f)$	range of f
$f^{(-1)}$	pseudo-inverse function to f
$\langle x, y \rangle$	inner product of x and y
$\|x\|$	norm of x
$d(x, y)$	distance between x and y
$B(x, \delta)$	open ball with center x and radius δ
$[0, 1]$	unit interval in $\mathbf{R}$
$\mathcal{C}(S)$	complement of S
$\mathrm{Ker}(X)$	kernel of X
$\mathrm{Ker}^*(X)$	strong kernel of X
$\mathrm{Ker}_\infty(X)$	co-kernel of X
$\mathrm{Ker}^*_\infty(X)$	strong co-kernel of X
$\mathrm{Int}(S)$	interior of S
$\mathrm{Rlint}(S)$	relative interior of S
$\mathrm{Cl}(S)$	closure of S
$\mathrm{Bd}(S)$	boundary of S
$U(f, \alpha)$	upper level set of f at α
$L(f, \alpha)$	lower level set of f at α
$H(f, \alpha)$	level set of f at α
$\mathrm{Epi}(f)$	epigraph of f
$\mathrm{Hyp}(f)$	hypograph of f
$\mathbf{I}(x, y)$	line segment joining x and y
$\mathbf{L}(x, y)$	line going through x and y
$\mathbf{H}(x, y)$	half line emanating from x through y
$\mathrm{Conv}(S)$	convex hull of S
$\dim(S)$	dimension of S
$\mathrm{Card}(S)$	cardinality of S, number of elements of S
$\mathrm{Ext}(S)$	set of all extreme points of S
$\mathrm{Core}(\mu)$	core of μ
$\mathrm{Supp}(\mu)$	support of μ
$\nabla f(x)$	gradient vector of f at x
$\nabla^2 f(x)$	Hessian matrix of f at x
T_M, S_M	minimum t-norm, maximum t-conorm
T_P, S_P	product t-norm, probabilistic sum t-conorm
T_L, S_L	Łukasiewicz t-norm, bounded sum t-conorm
T_D, S_D	drastic product t-norm, drastic sum t-conorm
OS^k	k-order statistic aggregation operator
OWA^n_W	order weighted averaging operator of dimension n
$[A]_\alpha$	α-cut of a fuzzy set A
$\mathcal{F}(X)$	set of all fuzzy subsets of X
$\mathcal{C}_N A$	complement of fuzzy set A with respect to negation N
$\mu_{\tilde{R}T}(A, B)$	T-fuzzy extension of relation R of fuzzy sets A and B

Preface

A large number of decision making and optimization problems can be formulated as follows: Given a set of *feasible alternatives* and a binary relation *better than* for a consistent mutual comparison of alternatives, find the *best* alternative. As a rule, the set of feasible alternatives is specified by a number of conditions as a subset of a given underlying set. The underlying set is usually equipped with some mathematical structure that can be more or less helpful in searching for the best feasible alternative. For almost all parts of this book, the underlying set is a finite-dimensional Euclidean space.

In a typical deterministic framework, the binary relation enabling comparison of alternatives is represented by a real-valued function f mapping the set of feasible alternatives into the set of real numbers in such a manner that a feasible alternative x is better than a feasible alternative y if and only if $f(x) > f(y)$ or $f(x) < f(y)$, one of these possibilities selected. In the former case the problem of finding the best alternative becomes that of maximizing f over the set of feasible alternatives; in the latter case, minimization of f is required. Construction of such an objective function may be a nontrivial task. Moreover, for some practically relevant binary relations such representations do not exist. It is therefore sometimes preferable or necessary to represent some relations by means of several functions. The meaning of maximization or minimization with respect of several real-valued functions may vary according to the underlying binary relation. For example: in some situations, the decision maker can be interested in finding a Pareto maximizer; in other situations, his wish may be to find a compromise solution.

Convexity of sets in linear spaces, and concavity and convexity of functions lie at the root of beautiful theoretical results that are at the same time extremely useful in the analysis and solution optimization problems, regardless of whether the optimization is required with respect to a single objective or multiple objectives. Fortunately, not each of these results relies necessarily on convexity or concavity. Some of them, for example the results guaranteeing

that each local optimum is also a global optimum, can be derived for substantially wider classes of problems. A large portion of the first part of this book is concerned with several types of generalized convex sets and generalized concave functions. In addition to their applicability to nonconvex optimization, they are used in the second part, where decision making and optimization problems under uncertainty are investigated.

Uncertainty in the problem data often cannot be avoided when dealing with practical problems. It may arise from errors in measuring physical quantities, from errors caused by representing some data in a computer, from the fact that some data are approximate solutions of other problems or estimations by human experts, etc. Over the last thirty years, the fuzzy set approach proved to be useful in some of these situations. It is this approach to optimization under uncertainty that is extensively used and studied in the second part of this book.

Usually, the membership functions of fuzzy sets involved in such problems are neither concave nor convex. They are, however, often quasiconcave or concave in some generalized sense. This opens possibilities for application of results on generalized concavity to fuzzy optimization. Interestingly, despite of this obvious relation, the interaction between these two areas has been rather limited so far. It is hoped that the presented combination of ideas and results from the field of generalized concavity on the one hand and fuzzy optimization on the other hand will be of interest to both communities and will result in an enlargement of the class of problems that can be satisfactorily handled.

In Chapter 1, for reader's convenience, we review some basic notation and concepts necessary for understanding of the text, particularly, we present some introductory elements of linear algebra and calculus.

In Chapter 2, we deal with generalized convex sets. The most natural generalization of convex sets are starshaped sets. As further generalization of starshaped sets we obtain path-connected sets, invex sets and univex sets. Finally, we introduce a new class of generalized convex sets.

Chapter 3 begins with the classical definitions of concave and quasiconcave functions. Then we introduce starshaped functions, quasiconnected functions, and a concavity of functions with respect to suitable sets of mappings and functions. This approach enable us to cover several other ways of introducing generalized concavity known from the literature. Differentiable generalized concave functions are also studied and mutual relationships among different classes of functions are presented. An application to constrained optimization is discussed.

In Chapter 4, we deal with functions defined on the n-dimensional Euclidean space $\mathbf{R}^n$ and having their values in the unit interval $[0, 1]$ of real numbers. Such functions naturally appear in optimization under uncertainty where they can be interpreted as membership functions of fuzzy subsets of $\mathbf{R}^n$ or possibility distributions, etc. Using the notions and properties of trian-

gular norms and triangular conorms, we introduce and study new classes of functions related to concavity. We call them (Φ, T)-concave functions because they are defined by choosing a class Φ of suitable mappings and a triangular norm T. We focus on deriving conditions under which local maximizers of such functions are also global maximizers.

In Chapter 5, we prepare a material for further study of multi-objective optimization problems by analyzing some properties of general aggregation operators applied to criteria in the form of generalized concave functions. We look for conditions guaranteeing some attractive local-global properties of aggregating mappings. In this context we review mainly well known classes of averaging aggregation operators: compensative operators, order-statistic operators and OWA operators. We present also general classes of aggregation operators generated by Sugeno and Choquet integrals.

In Chapter 6, we deal with fuzzy sets. Already in the early stages of the development of fuzzy set theory, it has been recognized that fuzzy sets can be defined and represented in several different ways. We define fuzzy sets within the classical set theory by nested families of sets, and then we discuss how this concept is related to the usual definition by membership functions. Binary and valued relations are extended to fuzzy relations and their properties are extensively investigated. Moreover, fuzzy extensions of real functions are studied, particularly, the problem of establishing sufficient conditions under which the membership function of the function value is quasiconcave. Sufficient conditions for commuting the diagram "mapping - α-cutting" is presented in the form of classical Nguyen's result.

In the second part of this book, we are concerned with applications of the theory presented in the first part.

In Chapter 7, we consider the problem to find a "best" decision in the set of feasible decisions with respect to several criteria functions. Within the framework of such a decision situation, we deal with the existence and mutual relationships of three kinds of "optimal decisions": Weak Pareto-Maximizers, Pareto-Maximizers and Strong Pareto-Maximizers - particular alternatives satisfying some natural and rational conditions. We study also the compromise decisions maximizing some aggregation of the criteria. The criteria considered will be functions defined on the set of feasible decisions with the values in the unit interval. As mentioned above such functions can be interpreted as membership functions of fuzzy subsets and will be called fuzzy criteria. Later on, in Chapters 8 and 9, each constraint or objective function of the fuzzy mathematical programming problem will be associated with a unique fuzzy criterion. From this point of view, Chapter 7 could follow the Chapters 8 and 9, which deal with fuzzy mathematical programming. Our approach is, however, more general and can be adopted to a more general class of decision problems. The

results of Chapter 5 are extended and presented in the framework of multi-criteria decision making.

Fuzzy mathematical programming problems (FMP) investigated in Chapter 8 form a subclass of decision-making problems where preferences between alternatives are described by means of objective functions defined on the set of alternatives in such a way that greater values of the functions correspond to more preferable alternatives. The values of the objective function describe effects from choices of the alternatives. The chapter begins with the formulation of a FMP problem associated with the classical mathematical programming problem (MP). After that we define a feasible solution of FMP problem and optimal solution of FMP problem as special fuzzy sets. From practical point of view, α-cuts of these fuzzy sets are important, particularly the nonempty α-cuts with the maximal α. Among others we show that the class of all MP problems with (crisp) parameters can be naturally embedded into the class of FMP problems with fuzzy parameters.

In Chapter 9, we deal with a class of fuzzy linear programming problems (FLP) and again introduce the concepts of feasible and optimal solutions - the necessary tools for dealing with such problems. In this way we show that the class of classical linear programming problems (LP) can be embedded into the class of FLP problems. Moreover, for FLP problems we define the concept of duality and prove the weak and strong duality theorems. Furthermore, we investigate special classes of FLP - interval LP problems, flexible LP problems, LP problems with interactive coefficients and LP problems with centered coefficients.

In Chapter 10, we first recall elementary concepts and basic models of deterministic machine scheduling. Then we discuss some of them in nondeterministic situations. We present motivation examples characterizing difficulties that may occur under uncertainty of problem parameters. Then we investigate some particular fuzzy models with fuzzy due dates, fuzzy processing times and fuzzy precedence relations. Finally we discuss some directions of the future research in the area of fuzzy machine scheduling.

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A famous theorem says that there is an error or misprint in every mathematical text longer than n pages where n is a small natural number. We have tried to minimize them but as no theorem can be false, we will be grateful for receiving information on errors, as well as comments and suggestions for improvement.

I

THEORY

Pure thought is always better than thoughtless counting

— Paul Halmos and Steven Givent
in *Logic as Algebra*

Chapter 1

PRELIMINARIES

We assume that the reader is familiar with standard set theoretic concepts, introductory elements of linear algebra, and basic material from calculus. To avoid misunderstandings and for reader's convenience, we review some basic concepts, results and notations.

If a is an element of a set A, we write $a \in A$; if not, we write $a \notin A$. When A is a *subset* of B (proper or improper), we write $A \subset B$. The *union* and the *intersection* of sets A and B is denoted by $A \cup B$ and $A \cap B$, respectively. The empty set is denoted by $\emptyset$, and the symbol $A \setminus B$ stands for the set of elements of A that do not belong to B.

A *binary relation* on a set A is a subset of the Cartesian product $A \times A$. If $\preceq$ is a binary relation on A and (a, b) belongs to $\preceq$, then we usually write $a \preceq b$ instead of $(a, b) \in \preceq$.

A binary relation $\preceq$ on A is called

(i) *reflexive* if $a \preceq a$ for each $a \in A$,

(ii) *antisymmetric* if $a \preceq b$ and $b \preceq a$ imply $a = b$ for all $a, b \in A$,

(iii) *transitive* if $a \preceq b$ and $b \preceq c$ imply $a \preceq c$ for all $a, b, c \in A$,

(iv) *linear* if $a \preceq b$ or $b \preceq a$ for all $a, b \in A$.

If a binary relation $\preceq$ on A satisfies (i) - (iii), then it is called a *partial ordering* relation on A. A partial ordering relation on A satisfying (iv) is called a *linear ordering* relation on A.

Let $\preceq$ be a partial ordering relation on a set X and let A be a subset of X. Then $x \in X$ is an *upper bound* of A if $a \preceq x$ for all $a \in A$. An upper bound u of A is called the *supremum* of A if, for each upper bound x of A, we have $u \preceq x$. The supremum of A is called *maximum* of A if it belongs to A.

We write $u = \sup A$ if u is the supremum of A, and $u = \max A$ if u is the maximum of A. Analogously, an element $x \in X$ is a *lower bound* of A if $x \preceq a$ for all $a \in A$. A lower bound u of A is the *infimum* of A if $x \preceq u$ for each lower bound x of A. We write $u = \inf A$ if u is the infimum of A. If the infimum u of A belongs to A, then we call it *minimum* of A and write $u = \min A$.

A set L with a partial ordering relation $\preceq$ is a lattice, if $\inf\{x, y\}$ and $\sup\{x, y\}$ exist for all $x, y \in L$.

Our working space throughout this book is the set $\mathbf{R}^n$ of ordered n-tuples of real numbers equipped with the structure of a *real vector space* and the standard *Euclidean* structure.

The former means that if λ is a real number (called also a scalar) and $x = (\xi_1, \xi_2, \ldots, \xi_n)$, $y = (\eta_1, \eta_2, \ldots, \eta_n)$ are from $\mathbf{R}^n$, then the multiplication of x by λ and the summation of x and y are defined by

$$\lambda x = (\lambda\xi_1, \lambda\xi_2, \ldots, \lambda\xi_n)$$

and

$$x + y = (\xi_1 + \eta_1, \xi_2 + \eta_2, \ldots, \xi_n + \eta_n).$$

The latter means that $\mathbf{R}^n$ is equipped with the *inner (scalar) product* $\langle x, y \rangle$ defined by

$$\langle x, y \rangle = \xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_n\eta_n.$$

It should be pointed out that this vector space can also be considered as an *affine space* with specified origin. As a consequence, elements of $\mathbf{R}^n$ can be called *vectors* or *points* depending on which structure is more appropriate or emphasized. Similarly, the numbers $\xi_1, \xi_2, \ldots, \xi_n$ from which $x = (\xi_1, \xi_2, \ldots, \xi_n)$ is composed are called *components* or *coordinates* of x. The vector whose all components are zero is denoted θ and called the *zero vector*.

Let us now recall without proofs some basic concepts and facts concerning the vector and affine space structures of $\mathbf{R}^n$. An element x of $\mathbf{R}^n$ is a *linear combination of a nonempty subset* A of $\mathbf{R}^n$, if $x = \sum_{i=1}^m \lambda_i a_i$ for some positive integer m, real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ and elements $a_1, a_2, \ldots, a_m$ of A. The empty set has, by definition, only one linear combination, namely, the zero vector. Depending on additional restrictions on $\lambda_1, \lambda_2, \ldots, \lambda_m$, we obtain various classes of linear combinations. The following ones are used extensively in the book.

A linear combination $\sum_{i=1}^m \lambda_i a_i$ is called

- an *affine combination*, if $\sum_{i=1}^m \lambda_i = 1$;
- a *convex combination*, if $\sum_{i=1}^m \lambda_i = 1$ and $\lambda_i \geq 0$ for $i = 1, 2, \ldots, m$;
- a *convex conical combination*, if $\lambda_i \geq 0$ for $i = 1, 2, \ldots, m$.

By somewhat misusing the language, we often write "a combination of elements" instead of "a combination of a set".

A subset S of $\mathbf{R}^n$ is called *linear (affine, convex)*, if S contains all its linear (affine, convex) combinations. Linear subsets are also called *(linear) subspaces* of $\mathbf{R}^n$, and affine subsets are also called *affine subspaces*. The intersection of all linear (affine, convex) sets of $\mathbf{R}^n$ containing a subset S of $\mathbf{R}^n$ is called a *linear (affine, convex) hull* of S. It turns out that a linear (affine, convex) hull of S is equal to the set of all linear (affine, convex) combinations of S, and is also the smallest, with respect to inclusion, linear (affine, convex) set containing S. The convex hull of a set S is denoted by $\text{Conv}(S)$.

If x and y are points of $\mathbf{R}^n$, then the convex hull of the set $\{x, y\}$ is called the *line segment* joining x and y. The line segments are denoted by $\mathbf{I}(x, y)$. Obviously

$$\mathbf{I}(x,y) = \{z \in \mathbf{R}^n \mid z = x + \lambda(y - x),\ \lambda \in [0,1]\}.$$

It is an easy exercise to show that a subset S of $\mathbf{R}^n$ is convex if and only if $\mathbf{I}(x, y) \subset S$ whenever $x, y \in S$.

If x and y are two different points of $\mathbf{R}^n$, then the affine hull of the set $\{x, y\}$ is called the *line going through x and y* and is denoted by $\mathbf{L}(x, y)$. Obviously

$$\mathbf{L}(x,y) = \{z \in \mathbf{R}^n \mid z = x + \lambda(y - x),\ \lambda \in \mathbf{R}\}.$$

The *half line emanating from x through y* is the set of all points of the form $x + \lambda(y - x)$, where $\lambda \geq 0$, that is,

$$\mathbf{H}(x,y) = \{z \in \mathbf{R}^n \mid z = x + \lambda(y - x),\ \lambda \geq 0\}.$$

An important concept for study of sets in vector spaces is that of *linear independence*. A vector x is said to be *linearly dependent on a set S*, if it can be expressed as a linear combination of vectors from S. If x is not linearly dependent on S, then it is said to be *linearly independent of S*. A set U containing at least two vectors is said to be a *linearly independent set*, if each vector $x \in U$ is linearly independent of $U \setminus \{x\}$. By convention, the empty set and the sets consisting only of a single nonzero vector are linearly independent; the set consisting only of the zero vector is a linearly dependent set. Often we say linearly dependent or independent vectors instead of linearly dependent or independent sets of vectors. Obviously vectors $x_1, x_2, \ldots, x_m$ are linearly independent if and only if the equality $\sum_{i=1}^m \lambda_i x_i = \theta$ implies $\lambda_i = 0$ for all $i = 1, 2, \ldots, m$.

A subset S of linearly independent vectors from a subspace L of $\mathbf{R}^n$ is said to be a *basis* for L, if S generates L in the sense that L is the linear hull of S. Every subspace of $\mathbf{R}^n$ has a basis and any two bases of a given subspace have the same number of elements; this number is called the *dimension* of the subspace.

The set $\{e_1, e_2, \ldots, e_n\}$ of vectors

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0), \\ e_2 &= (0, 1, 0, \ldots, 0), \\ &\vdots \\ e_n &= (0, 0, 0, \ldots, 1), \end{aligned}$$

is a basis of $\mathbf{R}^n$. This basis is called the canonical basis of $\mathbf{R}^n$. It follows that the dimension of $\mathbf{R}^n$ is n.

The operation of multiplication of a vector by a scalar λ and addition of vectors are extended to subsets A and B of vectors by

$$\lambda A = \{x \in \mathbf{R}^n \mid x = \lambda a,\ a \in A\},$$

and

$$A + B = \{x \in \mathbf{R}^n \mid x = a + b,\ a \in A,\ b \in B\}.$$

If A consists of a single vector a, that is, $A = \{a\}$, then we write $a + B$ instead of $\{a\} + B$ and say that the set $a + B$ is a *translate* of B by the vector a. Obviously $A + B$ can be interpreted as the union of all translates $a + B$ of B for $a \in A$. It turns out that every affine subspace A of $\mathbf{R}^n$ can be expressed as a translate $a + L$ of a linear subspace of $\mathbf{R}^n$. Under the dimension of A, we understand the dimension of the corresponding linear subspace. The dimension of the arbitrary nonempty subset of $\mathbf{R}^n$ is then defined as the dimension of its affine hull.

Now we turn our attention to topological concepts induced by the inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

First we notice that $\mathbf{R}^n$ can be considered as a norm space, or as a metric space by defining the Euclidean *norm* $\|x\|$ of a vector x and the Euclidean *distance* $d(x, y)$ from a point x to a point y as follows:

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}} \text{ and } d(x, y) = \|x - y\|.$$

Observe that $d(x, y)$ is the usual distance from $x = (\xi_1, \xi_2, \ldots, \xi_n)$ to $y = (\eta_1, \eta_2, \ldots, \eta_n)$, because

$$d(x, y) = \|x - y\| = \langle x - y, x - y \rangle^{\frac{1}{2}} = \left[\sum_{i=1}^{n} (\xi_i - \eta_i)^2\right]^{\frac{1}{2}}.$$

Having the concept of a norm or a distance at our disposal, we define the *standard topology* of $\mathbf{R}^n$ as follows. First we define, for each $x \in \mathbf{R}^n$ and $\delta > 0$, the *open ball* $B(x, \delta)$ centered at x with radius δ by

$$B(x, \delta) = \{y \in \mathbf{R}^n \mid d(x, y) < \delta\}.$$

Then we say that a point $x \in \mathbf{R}^n$ is an *interior point* of the set $S \subset \mathbf{R}^n$, if there is a $\delta > 0$ such that $B(x, \delta) \subset S$. Next we say that a set S is *open* if each of its points is an interior point of S.

The family of all open subsets of $\mathbf{R}^n$ defined in this manner is called the standard topology for $\mathbf{R}^n$. If S is a nonempty subset of $\mathbf{R}^n$, then the topology relative to S is the family of all sets U such that $U = S \cap V$ for some open set V of $\mathbf{R}^n$.

The *interior* of a set S, denoted by $\text{Int}(S)$, is the union of all the open subsets of S. Obviously the interior of S is the set of all interior points of S.

The *relative interior* of a set S, denoted by $\text{Rlint}(S)$, is the union of all the open sets in the relative topology to S.

A set S is *closed* if its *complement* $\mathcal{C}(S) = \mathbf{R}^n \setminus S$ is open. The *closure* $\text{Cl}(S)$ of S is the intersection of all the closed sets containing S. Also, a point x is in $\text{Cl}(S)$ if and only if, for every $\delta > 0$, the open ball $B(x, \delta)$ contains at least one point of S.

The *boundary* $\text{Bd}(S)$ of a set S is defined by $\text{Bd}(S) = \text{Cl}(S) \cap \text{Cl}(\mathcal{C}(S))$. A set S is *bounded* if there exists $\delta > 0$ such that $S \subset B(x, \delta)$. A closed set S is said to be *regular* if $\text{Cl}(\text{Int}(S)) = S$. A set S is said to be *compact* if it is closed and bounded.

Two sets A and B are *separated* if both $A \cap \text{Cl}(B)$ and $\text{Cl}(A) \cap B$ are empty. A set S is *connected* if S is not a union of two nonempty separated sets.

An infinite sequence $\{x_k\}_{k=1}^{\infty}$ of points in $\mathbf{R}^n$ is said to *converge* to the limit x, if the sequence $\{d(x_k, x)\}_{k=1}^{\infty}$ of numbers converges to zero as $k \longrightarrow +\infty$. If $\{x_k\}_{k=1}^{\infty}$ converges to x, we write $x_k \longrightarrow x$ or $\lim x_k = x$. Important observations are that if a sequence converges, then its limit is unique and that a sequence converges if and only if each component converges. It is also useful to notice that a set S is closed if and only if every convergent sequence of points from S has its limit in S.

Finally, we recall some notation and concepts related to real-valued functions defined on subsets of $\mathbf{R}^n$. Let α be a real number and X be a subset of $\mathbf{R}^n$. For a function $f : X \longrightarrow \mathbf{R}$, we find useful to introduce the following sets:

- The set
$$U(f, \alpha) = \{x \in X \mid f(x) \geq \alpha\}$$
is called *upper-level set* of f at α.

- The set
$$L(f, \alpha) = \{x \in X \mid f(x) \leq \alpha\}$$

is called *lower-level set* of f at α.

- The set
$$H(f, \alpha) = \{x \in X \mid f(x) = \alpha\}$$
is called *level set* of f at α.

- The set
$$\text{Epi}(f) = \{(x, t) \in X \times \mathbf{R} \mid f(x) \leq t\}$$
is called *epigraph* of f.

- The set
$$\text{Hyp}(f) = \{(x, t) \in X \times \mathbf{R} \mid f(x) \geq t\}$$
is called *hypograph* of f.

Also the following properties of real-valued functions will be used in the sequel. A function $f : X \to \mathbf{R}$ is said to be

- *upper semicontinuous* on X (USC on X) if all its upper-level sets $U(f, \alpha)$ are closed.

- *lower semicontinuous* on X (LSC on X) if all its lower-level sets $L(f, \alpha)$ are closed.

- *continuous* on X if it is both upper and lower continuous on X.

Since we are interested in optimization problems, the following well known fact is of great importance.

THEOREM 1.1 *Let X be a nonempty compact subset of $\mathbf{R}^n$. A function $f : \mathbf{R}^n \to \mathbf{R}$ upper semicontinuous (lower semicontinuous) on X attains its maximum (minimum) on X.*

As a consequence, each continuous function on a nonempty compact set X attains both its maximum and its minimum on X.

Chapter 2

GENERALIZED CONVEX SETS

In this chapter we first generalize convex sets by introducing the concept of starshaped sets, which is "one-step" more general than the concept of convex sets. Then we make further steps generalizing the starshaped sets to obtain path-connected sets, Φ-convex sets and some other types of generalized convex sets. The generalized convex sets will serve as a basis for defining generalized concave and convex functions, which will be introduced in the third chapter. We also pay attention to the separability of starshaped sets.

1. Convex Sets

Having surveyed the basic structures of $\mathbf{R}^n$, we now direct our attention to the basic ideas of convex sets. As mentioned in the introduction the notion of a convex set can be defined as follows.

DEFINITION 2.1 *A subset X of $\mathbf{R}^n$ is a* convex set *if, for every $x, y \in X$, we have $\mathbf{I}(x, y) \subset X$.*

Geometrically, a set is convex if with the end points belonging to X the whole line segment joining the end points lies in X. The following theorem is basic in the study of convex sets. It was first published by Caratheodory in 1907. The proof is elementary and can be found in [67].

THEOREM 2.2 (Caratheodory) *If S is a nonempty subset of $\mathbf{R}^n$, then every $x \in \mathrm{Conv}(S)$ can be expressed as a convex combination of $n + 1$ or fewer points of S.*

Let S be a convex set. A point $x \in S$ is called an *extreme point of S*, if $S \setminus \{x\}$ is convex. The set of all extreme points of S is denoted by $\mathrm{Ext}(S)$. It is a well known fact that nonempty every compact convex subset S of $\mathbf{R}^n$ is

the convex hull of its all extreme points, i.e.,

$$S = \mathrm{Conv}\,(\mathrm{Ext}(S))\,.$$

The convex hull of a finite set is called a *polyhedron.* The extreme point of a polyhedron S is called a *vertex of S.* If $S = \{x_1, x_2, \ldots, x_{k+1}\}$ and $\dim(S) = k$, then Conv(S) is called a *k-dimensional simplex.* Also the points $x_1, x_2, \ldots, x_{k+1}$ are vertices of the k-dimensional simplex Conv(S). Finally, we state a definition of strict convex sets.

DEFINITION 2.3 *A convex set $X \subset \mathbf{R}^n$ that contains at least two points is said to be* strictly convex *if, for every pair of different points x and y in the boundary of X, each point $\lambda x + (1 - \lambda)y$ with $\lambda \in (0,1)$ belongs to the interior of X.*

Notice that for each strictly convex set it holds Int$(X) \neq \emptyset$. We have also the following characterization of strictly convex sets; see [3].

PROPOSITION 2.4 *Let $X \subset \mathbf{R}^n$ be a convex set with* Int$(X) \neq \emptyset$. *Then X is strictly convex if and only if all its boundary points are extreme points.*

A detailed treatment of convex sets can be found in the monographs [130], [67].

2. Starshaped Sets

In the following definition we generalize the concept of convexity.

DEFINITION 2.5 *Let X be a subset of $\mathbf{R}^n$, $y \in X$. The set X is* starshaped from y *if, for every $x \in X$, we have* $\mathbf{I}(x, y) \subset X$. *The set of all points $y \in X$ such that X is starshaped from y is called the* kernel *of X and it is denoted by* Ker(X). *The set X is said to be a* starshaped set *if* Ker(X) *is nonempty, or X is empty.*

Clearly, X is starshaped if there is a point $y \in X$ such that X is starshaped from y. From the geometric viewpoint, if there exists a point y in X such that for every other point x from X the whole segment $\mathbf{I}(x, y)$ connecting the points x and y belongs to X, then X is starshaped. Evidently, every convex set is starshaped. For a convex set X, we have Ker$(X) = X$. Moreover, in the 1-dimensional space $\mathbf{R}$, convex sets and starshaped sets coincide. In the following examples we demonstrate that this conclusion is not true in $\mathbf{R}^2$.

EXAMPLE 2.6 *The Star set.* In Figure 2.1 (a) we have the non-convex starshaped set X_1 in $\mathbf{R}^2$. The dark shaded hexagonal K is the corresponding kernel. □

EXAMPLE 2.7 *The Moon set.* In Figure 2.1 (b) we have the non-convex star-shaped set X_2 in $\mathbf{R}^2$. The kernel is the singleton K. □

EXAMPLE 2.8 *The Twin set.* In Figure 2.1 (c) we have the non-convex star-shaped set X_3 in $\mathbf{R}^2$. The corresponding kernel is the segment K. □

EXAMPLE 2.9 *The Dirichlet set.* Let $f : [0, 2\pi) \to \{0, 1\}$ be the Dirichlet function, i.e. $f(\varphi) = 0$ if $\varphi \in [0, 2\pi)$ is a rational number and $f(\varphi) = 1$ if $\varphi \in [0, 2\pi)$ is an irrational number. Let $X_4 = \{(x, y) \in \mathbf{R}^2 \mid x = r\cos\varphi,\ y = r\sin\varphi,\ 0 \le r \le f(\varphi),\ 0 \le \varphi < 2\pi\}$, see Figure 2.1 (d). The corresponding kernel is the singleton K. Notice that X_4 is not convex and in every open ball centered at any point of X_4 there exist points not belonging to X_4. □

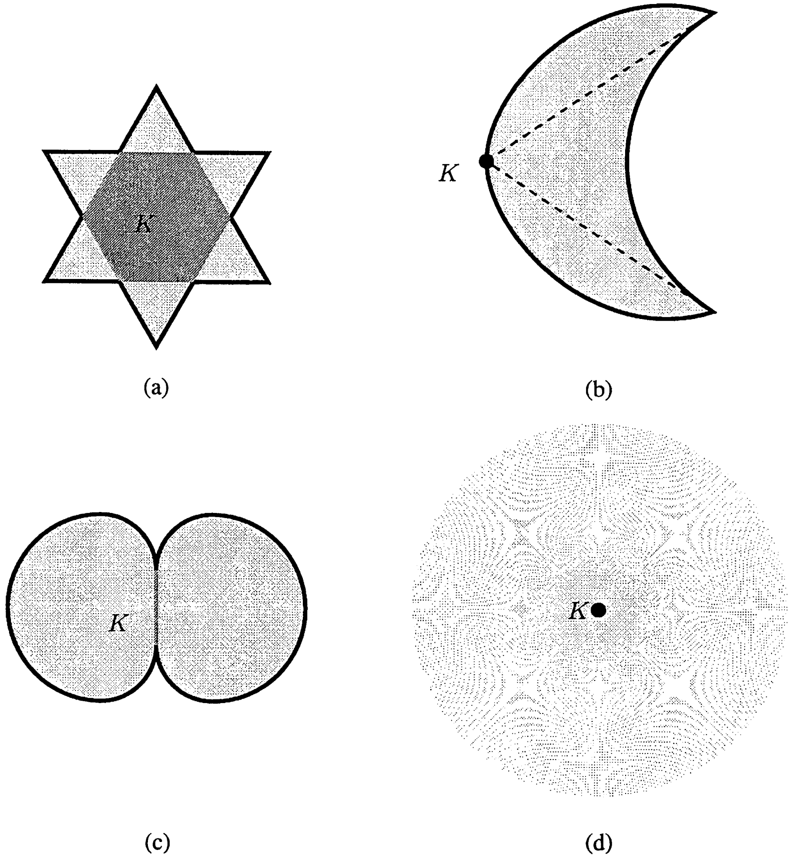

Figure 2.1.

EXAMPLE 2.10 In Figure 2.2 we have the convex (starshaped) set X_5 in $\mathbf{R}^2$ defined as follows: $X_5 = \{(x, y) \mid x \geq 0, y \geq 1/x\}$. Then X_6 defined as the complement of X_5 in $\mathbf{R}^2$, i.e., $X_6 = \mathbf{R}^2 \setminus X_5$, is starshaped; it is, however, a non-convex set. □

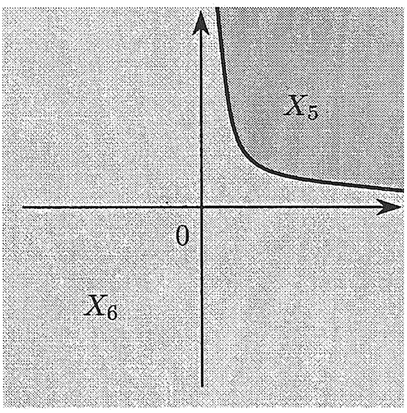

Figure 2.2.

Non-convex starshaped sets in $\mathbf{R}^3$, particularly starshaped polyhedra, have been known since the antiquity. According to the book [24] star polygons were first studied mathematically by Thomas Bradwardine (1290-1349) and were later investigated by the famous astronomer Johannes Kepler (1571-1630). A starshaped polyhedron can be obtained from a regular polyhedron by a process called *stellation.* This process is based on the idea of extending either faces of edges of the polyhedron and it was first described by Kepler in his book *Harmonices Mundi* published in 1619.

Further on, we shall investigate the properties of the starshaped sets. We begin with the well known results about kernels.

PROPOSITION 2.11 *Let X be a subset of $\mathbf{R}^n$. Then the kernel* $\mathrm{Ker}(X)$ *is convex.*

PROOF. Obviously, if X is not starshaped, then $\mathrm{Ker}(X)$ is empty, therefore convex.

Let X be starshaped, $x_1, x_2 \in \mathrm{Ker}(X)$ and $\lambda \in (0,1)$. By definition of $\mathrm{Ker}(X)$ we have $z = \lambda x_1 + (1 - \lambda)x_2 \in X$. We show that $z \in \mathrm{Ker}(X)$ by contradiction. Suppose the opposite, then there is a $u \in X$ and $\delta \in (0,1)$ such that $v = \delta u + (1 - \delta)z \notin X$. Since $u \in X$, $x_2 \in \mathrm{Ker}(X)$, we have $\varepsilon u + (1 - \varepsilon)x_2 \in X$ for every $\varepsilon \in (0,1)$. Then there exists $\varepsilon_0 \in (0,1)$ such

that for $w = \varepsilon_0 u + (1-\varepsilon_0)x_2$ we get $v = \gamma_0 w + (1-\gamma_0)x_1$ for some $\gamma_0 \in (0,1)$, see Figure 2.3. As $w \in X$, $x_1 \in \mathrm{Ker}(X)$, we have, by the definition of kernel, $v = \gamma_0 w + (1-\gamma_0)x_1 \in X$, a contradiction. ∎

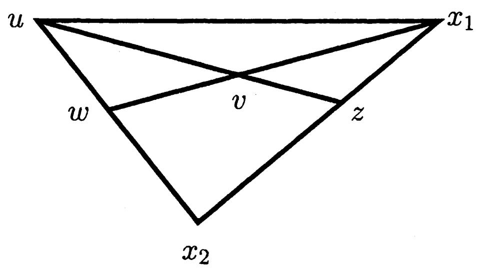

Figure 2.3.

PROPOSITION 2.12 *Let* $A \subset \mathbf{R}^n$, $B \subset \mathbf{R}^n$, $\alpha \in \mathbf{R}$. *Then*

$$\mathrm{Ker}(A) + \mathrm{Ker}(B) \subset \mathrm{Ker}(A+B) \tag{2.1}$$

and

$$\alpha\mathrm{Ker}(A) = \mathrm{Ker}(\alpha A). \tag{2.2}$$

PROOF. 1. Suppose that A and B are nonempty, otherwise the proof is trivial as $A + \emptyset = A$ for each $A \subset \mathbf{R}^n$. To prove inclusion (2.1), let $y \in \mathrm{Ker}(A) + \mathrm{Ker}(B)$. There exists $a \in \mathrm{Ker}(A)$, $b \in \mathrm{Ker}(B)$, with $y = a + b$. Taking arbitrary $x \in A + B$ and $\lambda \in (0,1)$, we show that $\lambda x + (1-\lambda)y \in A + B$. Indeed, there exists $u \in A$, $v \in B$ with $x = u + v$, such that $\lambda x + (1-\lambda)y = [\lambda u + (1-\lambda)a] + [\lambda v + (1-\lambda)b]$. The first term on the right side of the last equation belongs to A, whereas the second one belongs to B. Consequently, $\lambda x + (1-\lambda)y \in A + B$, thus $y \in \mathrm{Ker}(A+B)$.

2. Now we prove equality (2.2). If $y \in \alpha\mathrm{Ker}(A)$, then there is $u \in \mathrm{Ker}(A)$ with $y = \alpha u$. Taking arbitrary $x \in \alpha A$ and $\lambda \in (0,1)$, we show that $\lambda x + (1-\lambda)y \in \alpha A$.

There exists $v \in A$ with $x = \alpha v$ and then $\lambda x + (1-\lambda)y = \alpha[\lambda v + (1-\lambda)u]$, where the term in parentheses belongs to A, therefore $\lambda x + (1-\lambda)y \in \alpha A$. We have just proved

$$\alpha\mathrm{Ker}(A) \subset \mathrm{Ker}(\alpha A). \tag{2.3}$$

On the other hand, suppose that $y \in \mathrm{Ker}(\alpha A)$, then $y \in \alpha A$ and there is $u \in A$ with $y = \alpha u$. Without loss of generality we assume $\alpha \neq 0$. Using $\frac{1}{\alpha}$ instead

of α and αA instead of A in (2.3), we obtain $\frac{1}{\alpha}\text{Ker}(\alpha A) \subset \text{Ker}(\frac{1}{\alpha}\alpha A) = \text{Ker}(A)$ and therefore $u = \frac{1}{\alpha}y \in \frac{1}{\alpha}\text{Ker}(\alpha A) \subset \text{Ker}(A)$. Hence, $u \in \text{Ker}(A)$ and $y = \alpha u \in \alpha\text{Ker}(A)$. ■

PROPOSITION 2.13 *Let $A \subset \mathbf{R}^n$, $B \subset \mathbf{R}^n$, $\alpha \in \mathbf{R}$, A, B be starshaped. Then $A + B$ and αA are starshaped.*

PROOF. We show that $A + B$ is starshaped. As A, B are starshaped, there exists $y_1 \in \text{Ker}(A)$, and $y_2 \in \text{Ker}(B)$. By (2.1) $y_1 + y_2 \in \text{Ker}(A + B)$, thus $A + B$ is starshaped.

Similarly, since A is starshaped, there exists $y \in \text{Ker}(A)$, and by (2.2) $\alpha y \in \text{Ker}(\alpha A)$, thus αA is starshaped. ■

The following example shows that the opposite inclusion to (2.1) does not hold in general.

EXAMPLE 2.14 Define $A_1 = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1 = 0, x_2 \in [0, 1]\}$, $A_2 = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1 \in [0, 1], x_2 = 0\}$, $A = A_1 \cup A_2$, $B = \{x = (x_1, x_2) \in \mathbf{R}^2 \mid x_1 \in (0, 1), x_2 \in (0, 1)\}$. It can be verified that $\text{Ker}(A) = \{(0, 0)\}$, $\text{Ker}(B) = B$, consequently $\text{Ker}(A) + \text{Ker}(B) = B$. We can easily see that the point $(x_1, x_2) = (1, \frac{1}{2})$ belongs to $\text{Ker}(A + B)$, however, it does not belong to $B = \text{Ker}(A) + \text{Ker}(B)$. □

Now, we present some topological properties of starshaped sets. We shall see that some properties of starshaped sets are parallel to convex sets, some of them are, however, different.

Observe first that the interior of a starshaped set is not necessarily starshaped as it is evident from Example 2.7. In that example the interior $\text{Int}(X_2)$ of the Moon set X_2 is an open set with $\text{Ker}(\text{Int}(X_2)) = \emptyset$.

On the other hand, it follows from the definition of starshaped sets that any two points from a starshaped set can be connected by line segments with each point of the kernel, which implies connectedness of the starshaped set.

An intersection of starshaped (non-convex) sets does not need to be starshaped as the intersection may be disconnected. However, the following result holds.

PROPOSITION 2.15 *Let A_i, $i \in I$, be a family of starshaped sets, where I is an arbitrary set. Then*

$$\bigcap_{i \in I} \text{Ker}(A_i) \subset Ker\left(\bigcap_{i \in I} A_i\right).$$

PROOF. Let $x \in \bigcap_{i \in I} \text{Ker}(A_i)$, $y \in \bigcap_{i \in I} A_i$, $z = \lambda x + (1 - \lambda)y$ with $\lambda \in (0, 1)$. Since $\text{Ker}(A_i) \subset A_i$ for each $i \in I$ we have $x \in \bigcap_{i \in I} A_i$.

Moreover, for each $i \in I$ we have $x \in \text{Ker}(A_i)$ and $y \in A_i$, hence $z \in A_i$. Therefore, $z \in \bigcap_{i \in I} A_i$ and consequently $x \in Ker\left(\bigcap_{i \in I} A_i\right)$. ■

COROLLARY 2.16 *If* $\bigcap_{i \in I} \text{Ker}(A_i) \neq \emptyset$, *then* $\bigcap_{i \in I} A_i$ *is starshaped.*

The problem we deal now with is whether the closure of a starshaped set is starshaped. The answer is positive as it follows from the next proposition, which itself is an interesting result.

PROPOSITION 2.17 *If* $A \subset \mathbf{R}^n$ *is starshaped then*

$$\text{Cl}(\text{Ker}(A)) \subset \text{Ker}(\text{Cl}(A)). \tag{2.4}$$

PROOF. Let $x \in \text{Cl}(\text{Ker}(A))$. Then there exists a sequence x_k, $k = 1, 2, \ldots$, with $x_k \in \text{Ker}(A)$ and $x_k \to x$. We prove the proposition by contradiction. Assume that $x \notin \text{Ker}(\text{Cl}(A))$, then there exist $y \in \text{Cl}(A)$ and $\lambda \in (0, 1)$ such that $z = \lambda x + (1 - \lambda)y \notin \text{Cl}(A)$. Likewise, there exists a sequence y_k, $k = 1, 2, \ldots$, with $y_k \in A$ and $y_k \to y$. Setting $z_k = \lambda x_k + (1 - \lambda)y_k$, we obtain $z_k \in A$ for all $k = 1, 2, \ldots$, and $z_k \to z$. Consequently, $z \in \text{Cl}(A)$, a contradiction. ■

COROLLARY 2.18 *If* $A \subset \mathbf{R}^n$ *is closed and starshaped, then* $\text{Ker}(A)$ *is closed.*

PROOF. If A is closed then $\text{Cl}(A) = A$ and from (2.4) we obtain

$$\text{Cl}(\text{Ker}(A)) \subset \text{Ker}(A).$$

Hence $\text{Ker}(A)$ is closed. ■

COROLLARY 2.19 *If* $A \subset \mathbf{R}^n$ *is starshaped, then* $\text{Cl}(A)$ *is starshaped.*

PROOF. Let A be a nonempty starshaped set. Then $\text{Ker}(A)$ is nonempty. By (2.4) $\text{Ker}(\text{Cl}(A))$ is nonempty, thus $\text{Cl}(A)$ is starshaped. ■

The opposite inclusion to (2.4) is not true in general as the following example shows.

EXAMPLE 2.20 Define $A_1 = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1 \in (-1, 0), x_2 = 0\}$, $A_2 = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1 \in (0, 1), x_2 \in (0, 1)\}$, $A = A_1 \cup A_2$. We can see that $\text{Ker}(A) = \{(0, 0)\} = \text{Cl}(\text{Ker}(A))$ and $\text{Ker}(\text{Cl}(A)) = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1 \in [0, 1], x_2 = 0\}$, proving that $\text{Cl}(\text{Ker}(A)) \neq \text{Ker}(\text{Cl}(A))$. □

Caratheodory's Theorem (Theorem 2.2) provides a combinatorial characterization of convex sets. In some sense, an analogical result for starshaped sets was obtained by M. A. Krasnosselsky in 1946; see [130]. This theorem is popular under the name of "art gallery theorem": An art gallery consists of several connected rooms in which the walls are completely covered with pictures. The theorem implies that if for each three paintings in the gallery there exists a point from which all three can be seen, then there exists a point from which all the paintings in the gallery can be seen. This theorem is an interesting application of the theorem discovered by E. Helly as early as in 1913. The Helly's theorem, which is based on Caratheodory's Theorem 2.2, can be formulated as follows; see [67].

THEOREM 2.21 (Helly) *Let $\mathcal{F}$ be a family of compact convex subsets of* $\mathbf{R}^n$ *containing at least* $n+1$ *members. If every* $n+1$ *members of* $\mathcal{F}$ *have a point in common, then all the members of* $\mathcal{F}$ *have a point in common.*

More precisely stated, Krasnosselsky's theorem gives a Helly-type combinatorial criterion for determining whether or not a compact set is starshaped. The following definition will be useful.

Let S be a subset of $\mathbf{R}^n$. A point $y \in S$ is said to be *visible from a point* $x \in S$ *via* S if the segment $\mathbf{I}(x, y) \subset S$. The set of all points of S visible from $x \in S$ via S is called the x-*star of* S.

THEOREM 2.22 (Krasnosselsky) *Let* S *be a subset of* $\mathbf{R}^n$ *containing at least* $n+1$ *points. If for each* $n+1$ *points of* S *there is a point from which all* $n+1$ *points are visible, then the set* S *is starshaped.*

The simple proof of the theorem can be found in [67]. In Theorem 2.22, the assumption that "for each $n+1$ points of S there is a point from which all $n+1$ points are visible", can be weakened to the assumption that "for each $n+1$ *regular* points of S there is a point from which all $n+1$ points are visible". A point $y \in S$ is a regular point of S if there exists a closed halfspace which contains the y-star of S and which has y in the corresponding hyperplane. It is evident that a regular point of S is always a boundary point of S. For this and some other generalizations of Krasnosselsky and Helly's theorems; see [130].

3. Strongly Starshaped Sets

In this section we restrict our interest to a special class of starshaped sets having full dimension in $\mathbf{R}^n$.

DEFINITION 2.23 *Let* X *be a nonempty and closed subset of* $\mathbf{R}^n$, $y \in X$. *The set* X *is* strongly starshaped from y *if* $y \in \mathrm{Int}(X)$ *and for every* $x \in X$, *the half-line* $\mathrm{H}(y, x)$ *does not intersect the boundary* $\mathrm{Bd}(X)$ *more than once.*

The set of all points $y \in X$ *such that* X *is strongly starshaped from* y *is called a* strong kernel *of* X *and it is denoted by* $\mathrm{Ker}^*(X)$. *The set* X *is said to be a* strongly starshaped set *if* $\mathrm{Ker}^*(X)$ *is nonempty.*

PROPOSITION 2.24 *If* A *is a strongly starshaped subset of* $\mathbf{R}^n$, *then* A *is starshaped and*

$$\mathrm{Ker}^*(A) \subset \mathrm{Ker}(A). \tag{2.5}$$

PROOF. Let $y \in \mathrm{Ker}^*(A)$, $x \in A$, $x \neq y$ and $\lambda \in (0,1)$. We must show that $z = y + \lambda(x-y) \in A$. Assume the contrary, that is, $z = y + \lambda(x-y) \notin A$. Since A is closed, we have $\mathrm{Bd}(A) \subset A$ and the half-line $\mathbf{H}(y,x)$ intersects the boundary $\mathrm{Bd}(A)$ at $u' = y + t'(x-y)$ for some $0 < t' < \lambda$ and also at $u'' = y + t''(x-y)$ for some $\lambda < t''$, which contradicts the assumption that $\mathbf{H}(y,x)$ does not intersect the boundary $\mathrm{Bd}(A)$ more than once. Consequently, $y \in \mathrm{Ker}(A)$. ■

PROPOSITION 2.25 *If* A *is a closed starshaped subset of* $\mathbf{R}^n$ *such that* $\mathrm{Int}(\mathrm{Ker}(A)) \neq \emptyset$, *then* A *is strongly starshaped and*

$$Int\,(\mathrm{Ker}(A)) \subset \mathrm{Ker}^*(A). \tag{2.6}$$

PROOF. Let $y \in Int\,(\mathrm{Ker}(A))$, $x \in A$ and $x \neq y$. We show that the half-line $\mathbf{H}(y,x)$ does not intersect the boundary $\mathrm{Bd}(A)$ more than once. Assume the contrary, that is, the half-line $\mathbf{H}(y,x)$ intersects the boundary $\mathrm{Bd}(A)$ at $u' = y + t'(x-y)$ for some $0 < t'$ and also at $u'' = y + t''(x-y)$ for some $0 < t''$, where $t' < t''$.

Since $y \in Int\,(\mathrm{Ker}(A))$, we can find a ball $B(y,\varepsilon) \subset \mathrm{Ker}(A)$. Moreover, it is easy to demonstrate that there exists a sufficiently small $\delta > 0$ such that for any $v' \in B(u',\delta)$ and $v'' \in B(u'',\delta)$ with $v' \notin A, v'' \in A$ there exist $y' \in B(y,\varepsilon)$ and $\lambda \in (0,1)$ such that $v' = \lambda y' + (1-\lambda)v''$. But this is a clear contradiction with the fact that $y' \in \mathrm{Ker}(A)$. Thus, the half-line $\mathbf{H}(y,x)$ does not intersect the boundary $\mathrm{Bd}(A)$ in more than one point. ■

PROPOSITION 2.26 *If* A *is a strongly starshaped subset of* $\mathbf{R}^n$, *then* $\mathrm{Ker}^*(A)$ *is convex.*

PROOF. Let $x, y \in \mathrm{Ker}^*(A)$, $x \in A$ and $\lambda \in (0,1)$. We show that $z = y + \lambda(x-y) \in \mathrm{Ker}^*(A)$, particularly, we show that the half-line $\mathbf{H}(z,u)$ does not intersect the boundary $\mathrm{Bd}(A)$ more than once. Assume the contrary, that is, the half-line $\mathbf{H}(z,u)$ intersects the boundary $\mathrm{Bd}(A)$ at $u' = z + t'(u-z)$

for some $0 < t'$ and also at $u'' = z + t''(u - z)$ for some $0 < t''$, where $t' < t''$. Using a similar way of proof as was used in the preceding proposition, we obtain the required contradiction. ■

COROLLARY 2.27 *If A is a strongly starshaped subset of $\mathbf{R}^n$ such that* $\mathrm{Int}(\mathrm{Ker}(A)) \neq \emptyset$, *then*

$$Int\,(\mathrm{Ker}^*(A)) = \mathrm{Int}(\mathrm{Ker}(A)) \text{ and } Cl\,(\mathrm{Ker}^*(A)) = \mathrm{Ker}(A). \qquad (2.7)$$

PROOF. The first identity in (2.7) follows directly from (2.5) and (2.6). If A is strongly starshaped, then A is closed. By Corollary 2.18, $\mathrm{Ker}(A)$ is closed and by $Int\,(\mathrm{Ker}(A)) \neq \emptyset$, we obtain $\mathrm{Cl}(Int\,(\mathrm{Ker}^*(A))) = Cl\,(\mathrm{Ker}^*(A))$. Applying the first identity of (2.7), we obtain the second one. ■

Now, we formulate the counterpart of Proposition 2.12 for addition and multiplication of strongly starshaped sets.

PROPOSITION 2.28 *Let A be a strongly starshaped subset of $\mathbf{R}^n$ and X be a starshaped subset of $\mathbf{R}^n$.*

(i) *If $A + X$ is closed, then $A + X$ is strongly starshaped.*

(ii) *If $\lambda > 0$ then λA is strongly starshaped.*

PROOF. (i) Let $a \in \mathrm{Ker}^*(A)$, $x \in X$, $z = a + x$ and let u be an arbitrary point from $A + X$. We show that the half-line $\mathbf{H}(z, u)$ does not intersect the boundary $\mathrm{Bd}(A)$ more than once. On the contrary, suppose that the half-line $\mathbf{H}(z, u)$ intersects the boundary $\mathrm{Bd}(A + X)$ at $u' = z + t'(u - z)$ for some $0 < t'$ and also at $u'' = z + t''(u - z)$ for some $0 < t''$, where $t' < t''$. By a construction similar to the proof of Proposition 2.25, we obtain the desired contradiction. Hence, the first part of the proposition follows.

(ii) This part is a direct application of the definitions of starshaped and strongly starshaped sets. ■

4. Co-Starshaped Sets

In this section we study co-starshaped sets which are special sets in some sense complementary to starshaped ones.

DEFINITION 2.29 *Let X be a subset of $\mathbf{R}^n$, $y \notin X$. The set X is* co-starshaped from *y if for every $x \in X$, the set $\mathbf{H}(y, x) \setminus \mathbf{I}(x, y)$ is a subset of X. The set of all points $y \in \mathbf{R}^n$ such that X is co-starshaped from y is called a* co-kernel *of X and it is denoted by* $\mathrm{Ker}_\infty(X)$. *The set X is said to be a* co-starshaped set *if* $\mathrm{Ker}_\infty(X)$ *is nonempty.*

Note that in Definition 2.29, the set $\mathbf{H}(y,x) \setminus \mathbf{I}(x,y) = \{z \in \mathbf{R}^n \mid z = y + t(x-y), t > 1\}$.

DEFINITION 2.30 *Let X be a nonempty and closed subset of $\mathbf{R}^n$, $y \notin X$. The set X is* strongly co-starshaped from y *if, for every $x \in X$, $\mathbf{H}(y,x) \cap X$ is either empty or contains at least two points, and the half-line $\mathbf{H}(y,x)$ does not intersect the boundary* $\mathrm{Bd}(X)$ *more than once. The set of all points $y \in \mathbf{R}^n$ such that X is strongly co-starshaped from y is called a* strong co-kernel *of X and it is denoted by* $\mathrm{Ker}^*_\infty(X)$. *The set X is said to be a* strongly co-starshaped set *if* $\mathrm{Ker}^*_\infty(X)$ *is nonempty.*

By the definitions, each strongly co-starshaped set is co-starshaped. Moreover, if X is strongly co-starshaped, then $\mathrm{Ker}_\infty(X) \subset \mathrm{Ker}^*_\infty(X)$. The following proposition is useful. Its proof is straightforward and follows directly from the definitions of regularity, closedness and complementarity.

PROPOSITION 2.31 *If X is a regular subset of $\mathbf{R}^n$ and $x \in \mathbf{R}^n$, then*

$$Cl\,(\mathcal{C}\,(x+X)) = x + Cl\,(\mathcal{C}\,(X))\,.$$

■

COROLLARY 2.32 *A regular set is strongly starshaped if and only if the closure of its complement is strongly co-starshaped.*

COROLLARY 2.33 *A regular set is strongly co-starshaped if and only if the closure of its complement is strongly starshaped.*

EXAMPLE 2.34 Let X be a nonempty open convex set. Then $\mathrm{Int}(X) = \mathrm{Int}(\mathrm{Ker}(X)) \neq \emptyset$, and $\mathrm{Cl}(X)$ is regular and convex. By Proposition 2.25, X is strongly starshaped and by Corollary 2.32, $Cl\,(\mathcal{C}\,(X))$ is also strongly co-starshaped. □

5. Separation of Starshaped Sets

The separation property plays a crucial role in many applications of convexity. Roughly speaking, if the intersection of two convex sets is either empty or coincides with the intersection of their boundaries, then these sets can be separated by the level set of a nonzero linear function, or, geometrically, by a hyperplane. The separation property for two convex sets easily follows from the following simple property: If a point does not belong to a closed convex set, then this point can be separated from this set by a hyperplane. In case the point belongs to the boundary of the set, this property is called the supporting property. Generalizations of these concepts are studied in the framework of abstract convexity; see [25], [110].

In this section, after starting with standard theory of separation of two convex sets by a hyperplane, we present an interesting notion of separability of starshaped sets by finite families of hyperplanes and by finite families of linearly independent linear functionals; see [110], [112]. We also show that the separability by a finite family of linearly independent linear functions can be studied in the framework of cone separability. Both concepts of separation are suitable for separating of special non-convex sets, particularly starshaped and co-starshaped ones; see also [110], [112], [131].

5.1. Separating Hyperplanes

In this section we will look carefully at the problem of separation and then use our results to obtain a characterization of closed convex sets. The essential role is played here by the concept of a hyperplane which is a natural generalization of the concept of a line in $\mathbf{R}^2$ and a plane in $\mathbf{R}^3$.

DEFINITION 2.35 *A mapping $f : \mathbf{R}^n \to \mathbf{R}^m$ is said to be* linear *if it is* additive *and* homogeneous, *that is, for every $x, y \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$, it holds*

$$f(x+y) = f(x) + f(y)$$

and

$$f(\lambda x) = \lambda f(x).$$

If $m = 1$, then the mapping f is called a linear functional. *For a nonzero linear functional f and $\alpha \in \mathbf{R}$, the level set*

$$H(f,\alpha) = \{x \in \mathbf{R}^n \mid f(x) = \alpha\}$$

is called a hyperplane. *The upper level set $U(f,\alpha)$ and lower level set $L(f,\alpha)$, where*

$$\begin{aligned} U(f,\alpha) &= \{x \in \mathbf{R}^n \mid f(x) \geq \alpha\}, \\ L(f,\alpha) &= \{x \in \mathbf{R}^n \mid f(x) \leq \alpha\}, \end{aligned}$$

are called the closed half-spaces.

The following proposition gives a correspondence between linear functionals and vectors. A simple proof can be found in [67].

PROPOSITION 2.36 *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a linear functional. Then there exists a unique vector $u \in \mathbf{R}^n$ such that*

$$f(x) = \langle u, x \rangle$$

holds for all $x \in \mathbf{R}^n$.

COROLLARY 2.37 *Let H be a hyperplane in $\mathbf{R}^n$. Then there exists a vector $u \in \mathbf{R}^n$, $u \neq \theta$ and a number $\beta \in \mathbf{R}$, such that*

$$H = \{x \in \mathbf{R}^n \mid \langle u, x\rangle = \beta\}.$$

DEFINITION 2.38 *Let A, B be nonempty subsets of $\mathbf{R}^n$, let $f : \mathbf{R}^n \to \mathbf{R}$ be a nonzero linear functional, and $\alpha \in \mathbf{R}$. The hyperplane $H(f, \alpha)$* separates the sets A and B *if one of the following holds:*

(i) $f(x) \leq \alpha$ *for all* $x \in A$ *and* $f(y) \geq \alpha$ *for all* $y \in B$,

(ii) $f(x) \geq \alpha$ *for all* $x \in A$ *and* $f(y) \leq \alpha$ *for all* $y \in B$.

DEFINITION 2.39 *Let A, B be nonempty subsets of $\mathbf{R}^n$, let $f : \mathbf{R}^n \to \mathbf{R}$ be a nonzero linear functional, and $\alpha \in \mathbf{R}$. The hyperplane $H(f, \alpha)$* strictly separates the sets A and B *if one of the following holds:*

(i) $f(x) < \alpha$ *for all* $x \in A$ *and* $f(y) > \alpha$ *for all* $y \in B$,

(ii) $f(x) > \alpha$ *for all* $x \in A$ *and* $f(y) < \alpha$ *for all* $y \in B$.

DEFINITION 2.40 *Let A, B be nonempty subsets of $\mathbf{R}^n$. We say that A and B* are (strictly) separable *if there exists a hyperplane that (strictly) separates A and B.*

We notice that strict separation of two nonempty sets requires that the sets are disjoint, while mere separation does not. Although it is necessary that two sets should be disjoint in order to be strictly separable, this condition is not sufficient, even for closed convex sets. Thus the problem of the existence of a hyperplane (strictly) separating two sets is more complex than it might appear at first glance. The following theorems give the solution of the problem of (strict) separation. The corresponding proofs can be found in [67].

THEOREM 2.41 *Let A, B be nonempty convex subsets of $\mathbf{R}^n$, such that*

$$\mathrm{Int}(A) \neq \emptyset \textit{ and } \mathrm{Int}(A) \cap B = \emptyset.$$

Then the sets A and B are separable.

It is easy to see that the requirement that one of the sets must have a nonempty interior cannot be eliminated. However, it turns out that this requirement can be weakened. The next theorem gives the best possible characterization of a separation of two convex sets. Two immediate consequences follow.

THEOREM 2.42 *Let A and B be nonempty convex subsets of $\mathbf{R}^n$, such that $\dim(A \cup B) = n$. Then A and B are separable if and only if*

$$\mathrm{Rlint}(A) \cap \mathrm{Rlint}(B) = \emptyset. \tag{2.8}$$

COROLLARY 2.43 *Let A and B be nonempty convex subsets of $\mathbf{R}^n$. If (2.8) holds, then A and B are separable.*

COROLLARY 2.44 *Two disjoint nonempty convex subsets of $\mathbf{R}^n$ are separable.*

Now, we turn to the problem of strict separation of two nonempty convex sets by a hyperplane. It is easy to construct examples of two disjoint non-compact convex sets that cannot be strictly separated. However, if both sets are compact, then such a separation is always possible. In fact, the following, slightly stronger result, holds.

THEOREM 2.45 *Let A and B be nonempty convex subsets of $\mathbf{R}^n$ such that A is compact and B is closed. Then A and B are strictly separable if and only if A and B are disjoint.*

Several extensions of this theorem are of particular interest. For example, the following results are quite useful.

THEOREM 2.46 *Let A and B be nonempty compact subsets of $\mathbf{R}^n$. Then A and B are strictly separable if and only if*

$$\mathrm{Conv}(A) \cap \mathrm{Conv}(B) = \emptyset.$$

COROLLARY 2.47 *A point x can be strictly separated from a compact set A by a hyperplane if and only if $x \notin \mathrm{Conv}(A)$.*

COROLLARY 2.48 *A point x can be strictly separated from a compact set A by a hyperplane if and only if for each subset S of $n+1$ or fewer points of A, there exists a hyperplane strictly separating x and S.*

5.2. Separation by a Family of Hyperplanes

In this section we extend the concept of separation of two sets by a hyperplane to the concept of separation of two sets by the level set of the minimum of a finite family of linear functionals.

DEFINITION 2.49 *Let A and B be nonempty subsets of $\mathbf{R}^n$. The family $\mathcal{H} = \{H_i \mid H_i = H_i(f_i, \alpha_i),\ i = 1, 2, \ldots, m\}$ of hyperplanes* separates the sets A and B *if one of the following holds:*

(i) *for every pair $x \in A$ and $y \in B$ there exists $H_j \in \mathcal{H}$ such that*

$$\begin{aligned} f_j(x) &\leq \alpha_j, \\ \alpha_j &\leq f_j(y), \end{aligned}$$

(ii) *for every pair $x \in A$ and $y \in B$ there exists $H_j \in \mathcal{H}$ such that*

$$\begin{aligned} f_j(x) &\geq \alpha_j, \\ \alpha_j &\geq f_j(y). \end{aligned}$$

DEFINITION 2.50 *Let A and B be nonempty subsets of $\mathbf{R}^n$. The family $\mathcal{H} = \{H_i \mid H_i = H_i(f_i, \alpha_i),\ i = 1, 2, \ldots, m\}$ of hyperplanes* strictly separates the sets A and B *if one of the following holds:*

(i) *for every pair $x \in A$ and $y \in B$ there exists $H_j \in \mathcal{H}$ such that*

$$\begin{aligned} f_j(x) &< \alpha_j, \\ \alpha_j &< f_j(y), \end{aligned}$$

(ii) *for every pair $x \in A$ and $y \in B$ there exists $H_j \in \mathcal{H}$ such that*

$$\begin{aligned} f_j(x) &> \alpha_j, \\ \alpha_j &> f_j(y). \end{aligned}$$

DEFINITION 2.51 *Let A and B be nonempty subsets of $\mathbf{R}^n$. We say that A and B* are (strictly) separable by a finite family of hyperplanes *if there exists a finite family of hyperplanes that (strictly) separates A and B.*

If $m = 1$, i.e., if the family $\mathcal{H}$ contains only one hyperplane, then Definitions 2.38 and 2.39 are equivalent to Definitions 2.49 and 2.50, respectively. The following proposition gives a sufficient condition for separation of two sets by a finite family of hyperplanes.

PROPOSITION 2.52 *Let A and B be nonempty subsets of $\mathbf{R}^n$. If one of the following conditions* (i) *or* (ii) *holds, then A and B are separable by a finite family of hyperplanes:*

(i) *there exists a family of linear functionals $\{f_i \mid i = 1, 2, \ldots, m\}$ and $\alpha \in \mathbf{R}$ such that*

$$\min\{f_i(x) \mid i = 1, 2, \ldots, m\} \leq \alpha$$

for all $x \in A$ and

$$\min\{f_i(y) \mid i = 1, 2, \ldots, m\} \geq \alpha$$

for all $y \in B$;

(ii) *there exists a family of linear functionals $\{f_i \mid i = 1, 2, \ldots, m\}$ and $\alpha \in \mathbf{R}$ such that*

$$\min\{f_i(x) \mid i = 1, 2, \ldots, m\} \geq \alpha$$

for all $x \in A$ *and*

$$\min\{f_i(y) \mid i = 1, 2, \ldots, m\} \leq \alpha$$

for all $y \in B$.

PROOF. Let condition (i) be satisfied and let $x \in A$ and $y \in B$. If j is the index from $\{1, 2, \ldots, m\}$ where the minimum of $\{f_i(x) \mid i = 1, 2, \ldots, m\}$ is attained, we have $f_j(x) = \min\{f_i(x) \mid i = 1, 2, \ldots, m\}$. Consequently, $f_j(x) \leq \alpha$. Since for $y \in B$ we have $f_j(y) \geq \min\{f_i(y) \mid i = 1, 2, \ldots, m\} \geq \alpha$, then $\mathcal{H} = \{H_i \mid H_i = H_i(f_i, \alpha), i = 1, 2, \ldots, m\}$ is the family of hyperplanes we look for.

For condition (ii) the proof is analogous. ■

5.3. Separation by a Family of Linear Functionals

Proposition 2.52 leads to the following generalization of the concept of separation by a finite family of hyperplanes.

DEFINITION 2.53 *Let* A *and* B *be nonempty subsets of* $\mathbf{R}^n$. *The family* $\mathcal{F} = \{f_i \mid i = 1, 2, \ldots, m\}$ *of linear functional* $f_i : \mathbf{R}^n \to \mathbf{R}$ separates the sets A and B *if each* f_i *is nonzero and one of the following holds:*

(i) *for every pair* $x \in A$ *and* $y \in B$ *there exists* $f_j \in \mathcal{F}$ *such that*

$$f_j(x) \leq f_j(y);$$

(ii) *for every pair* $x \in A$ *and* $y \in B$ *there exists* $f_j \in \mathcal{F}$ *such that*

$$f_j(x) \geq f_j(y).$$

DEFINITION 2.54 *Let* A *and* B *be nonempty subsets of* $\mathbf{R}^n$. *The family* $\mathcal{F} = \{f_i \mid i = 1, 2, \ldots, m\}$ *of linear functionals* $f_i : \mathbf{R}^n \to \mathbf{R}$ strictly separates the sets A and B *if each* f_i *is nonzero and one of the following holds:*

(i) *for every pair* $x \in A$ *and* $y \in B$ *there exists* $f_j \in \mathcal{F}$ *such that*

$$f_j(x) < f_j(y);$$

(ii) *for every pair* $x \in A$ *and* $y \in B$ *there exists* $f_j \in \mathcal{F}$ *such that*

$$f_j(x) > f_j(y).$$

DEFINITION 2.55 *Let* A *and* B *be nonempty subsets of* $\mathbf{R}^n$. *We say that* A and B are (strictly) separable by a finite family of linear functionals *if there exists a finite family of linear functionals that (strictly) separates* A *and* B.

If $m = 1$, i.e., if the family $\mathcal{F}$ contains only one linear functional, then the concepts of separation by a hyperplane and separation by a linear functional coincide. For $m > 1$, the same statement is not true, see Example 2.56.

Clearly, if A and B are separable by a family of hyperplanes $\{H_i \mid H_i = H_i(f_i, \alpha),\ i = 1, 2, \dots, m\}$, then A and B are separable by a family of linear functionals $\{f_i \mid i = 1, 2, \dots, m\}$. The following examples show that the opposite assertion does not hold.

EXAMPLE 2.56 Let A and B be subsets of $\mathbf{R}$, $A = \{0\}$, $B = \{-1, 1\}$. It is obvious that A and B cannot be separated by any hyperplane. Let for every $x \in \mathbf{R}$, $f_1(x) = x$, $f_2(x) = -x$ and let $\alpha = 0$. Clearly, A and B are separable by the family of 2 hyperplanes $\{H_i \mid H_i = H_i(f_i, 0),\ i = 1, 2\}$. Hence, A and B are also separable by the family of 2 linear functionals $\{f_1, f_2\}$. □

EXAMPLE 2.57 Let A and B be arbitrary nonempty subsets of $\mathbf{R}$, let for every $x \in \mathbf{R}$, $f_1(x) = x$, $f_2(x) = -x$. Then for each pair $a \in A$ and $b \in B$ either $a \le b$ or $a \ge b$. In the former case, $f_1(a) \le f_1(b)$; in the later case we have $f_2(a) \le f_2(b)$. Hence, A and B are separable by the family of 2 linear functionals $\{f_1, f_2\}$. □

EXAMPLE 2.58 Let $A = \mathcal{I}$ and $B = \mathbf{R} \setminus \mathcal{I}$, where $\mathcal{I}$ is the set of all integers. It is clear that A and B are not separable by any finite family of hyperplanes, however, by Example 2.57, A and B are separable by the family of two linear functionals. □

The last two examples also show that there must be some additional restricting requirement on the separating family of linear functionals, otherwise every two subsets (even overlapping ones) could be separable by a finite family of linear functionals. The requirement of linear independence of linear functionals, introduced by the following definition, turns out to be suitable for this purpose. Also notice that the definition is based on Proposition 2.36, according to which for every nonzero linear functional $f : \mathbf{R}^n \to \mathbf{R}$ there exists a unique vector $u \in \mathbf{R}^n$ such that $f(x) = \langle u, x \rangle$ for all $x \in \mathbf{R}^n$.

DEFINITION 2.59 *Let $f_1, f_2, \dots, f_m$ be linear functionals on $\mathbf{R}^n$, and let $u_1, u_2, \dots, u_m$ be vectors such that $f_i(x) = \langle u_i, x \rangle$ holds for all $x \in \mathbf{R}^n$ and every $i = 1, 2, \dots, m$. We say that the linear functionals $f_1, \dots, f_m$ are* linearly independent *if the vectors $u_1, \dots, u_m$ are linearly independent. If the linear functionals $f_1, \dots, f_m$ are not linearly independent we say that they are* linearly dependent.

5.4. Separation by a Cone

We turn now to the problem of separating two sets by a cone.

DEFINITION 2.60 *A subset C of $\mathbf{R}^n$ is called a* cone *if for each $x \in C$ and each $\lambda > 0$ we have $\lambda x \in C$. A cone C is said to be* solid *if* $\mathrm{Int}(C) \neq \emptyset$. *Let S be a subset of $\mathbf{R}^n$. A subset*

$$Cone(S) = \{x \in \mathbf{R}^n \mid x = \lambda y,\, y \in S,\, \lambda > 0\}$$

is called a conic hull *of S.*

Evidently, a conic hull $Cone(S)$ of any subset S of $\mathbf{R}^n$ is a cone. If S is convex then $Cone(S)$ is a convex cone. It is well known (see e.g. [67]) that a subset C of $\mathbf{R}^n$ is a convex cone if and only if $C + C = C$.

DEFINITION 2.61 *Let A and B be nonempty subsets of $\mathbf{R}^n$, C be a convex solid cone. We say that the cone C* separates *the sets A and B if one of the following holds:*

(i) *there exists $y \in \mathbf{R}^n$ such that*

$$A \subset y + C \text{ and } (y + \mathrm{Int}(C)) \cap B = \emptyset.$$

(ii) *there exists $y \in \mathbf{R}^n$ such that*

$$B \subset y + C \text{ and } (y + \mathrm{Int}(C)) \cap A = \emptyset.$$

The sets A and B are said to be cone-separable *if there exists a convex solid cone C, which separates A and B.*

It is convenient to present cone separation in terms of separation by a finite family of linear functionals. Recall the correspondence between linear functionals and the inner product in Proposition 2.36 and consider m vectors $u_1, u_2, \ldots, u_m$ in $\mathbf{R}^n$. Let

$$\begin{aligned} C_m^< &= \{x \in \mathbf{R}^n \mid \langle u_j, x\rangle < 0, j = 1, 2, \ldots, m\}, \\ C_m^> &= \{x \in \mathbf{R}^n \mid \langle u_j, x\rangle > 0, j = 1, 2, \ldots, m\}. \end{aligned}$$

Clearly, both $C_m^<$ and $C_m^>$ are open convex cones. If $m \le n$ and u_j, $j = 1, 2, \ldots, m$, are linearly independent, then both $C_m^<$ and $C_m^>$ are nonempty.

PROPOSITION 2.62 *Let C be a solid cone in $\mathbf{R}^n$ and $y \in \mathrm{Int}(C)$. Then there exist n linearly independent vectors $u_1, u_2, \ldots, u_n$ in $\mathbf{R}^n$ such that $C_n^> \subset \mathrm{Int}(C)$ and $\langle u_j, y\rangle = 1$ for all $j \in J = \{1, 2, \ldots, n\}$.*

PROOF. Let $H = \{x \in \mathbf{R}^n \mid \langle y, x\rangle = 0\}$. Since $y \in \mathrm{Int}(C)$, it follows that there exists a simplex $S \subset H$ such that $\theta \in \mathrm{Rlint}(S)$ and $y + S \in \mathrm{Int}(C)$. Let $s_1, s_2, \ldots, s_n$ be vertices of S. Then $y + s_1, y + s_2, \ldots, y + s_n$ are the vertices of $y + S$. We now verify that vectors

$$y, y + s_1, y + s_2, \ldots, y + s_{k-1}, y + s_{k+1}, \ldots, y + s_n, \tag{2.9}$$

are linearly independent for each $k = 1, 2, \ldots, n$. Assume that there exist numbers α_i, $i = 1, 2, \ldots, n$ such that

$$\theta = \alpha_k y + \sum_{i \neq k} \alpha_i (y + s_i) = \left(\sum_{i=1}^{n} \alpha_i \right) y + \sum_{i \neq k} \alpha_i s_i. \tag{2.10}$$

Since y is orthogonal to H and $s_i \in H$, it follows from (2.10) that

$$\sum_{i=1}^{n} \alpha_i = 0. \tag{2.11}$$

Applying (2.10) once again, we obtain $\sum_{i \neq k} \alpha_i s_i = \theta$. Since $\theta \in \text{Rlint}(S)$, it follows that its vertices s_i are linearly independent, hence $\alpha_i = 0$ for all $i \in J$ and $i \neq k$. Combining this result with (2.11) we obtain $\alpha_k = 0$. Thus the vectors (2.9) are linearly independent. It follows that the system of linear equations

$$\langle u, y \rangle = 1,\ \langle u, y + s_i \rangle = 0,\ i \in J,\ i \neq k,$$

has the unique solution denoted by u_k.

Now, we check that $u_1, u_2, \ldots, u_n$ are linearly independent vectors. Since $\langle u_k, s_i \rangle = -1$ for all $i \neq k$, it follows that

$$H_k = \{x \in H \mid \langle u_k, x \rangle \leq 0\} \neq H.$$

Moreover, it follows that $\langle u_k, s_k \rangle > 0$, since otherwise $S \subset H_k$, which is impossible since θ is an interior point of S with respect to H.

Let β_k, $k \in J$, be numbers such that $\sum_{k=1}^{n} \beta_k u_k = \theta$. Then

$$\sum_{k=1}^{n} \beta_k \langle u_k, y \rangle = \sum_{k=1}^{n} \beta_k = 0.$$

Therefore, for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} \sum_{k=1}^{n} \beta_k \langle u_k, s_i \rangle &= \sum_{k \neq i} \beta_k \langle u_k, s_i \rangle + \beta_i \langle u_i, s_i \rangle \\ &= \beta_i \langle u_i, s_i \rangle - \sum_{k \neq i} \beta_k \\ &= \beta_i \langle u_i, s_i \rangle - \sum_{k=1}^{n} (\beta_k + \beta_i) \\ &= \beta_i \left(1 - \langle u_i, s_i \rangle\right). \end{aligned}$$

Since $\langle u_i, s_i \rangle \neq 1$, it follows that $\beta_i = 0$.

Thus, the set of linearly independent vectors $\{u_1, u_2, \ldots, u_n\}$ has the following property: for each k, all vertices of the simplex S excluding the vertex s_k belong to the hyperplane $\langle u_k, y\rangle = 1$ and s_k belongs to the halfspace $\langle u_k, y\rangle > 0$. Consequently, S can be represented in the following form:

$$S = \{x \mid \langle u_k, x\rangle \geq 1,\ k = 1, 2, \ldots, n,\ \langle y, x\rangle = 0\}.$$

Since $\langle u_k, y\rangle = 1$, we also have

$$y + S = \{x \mid \langle u_k, x\rangle \geq 0,\ k = 1, 2, \ldots, n,\ \langle y, x\rangle = \|y\|^2\}. \tag{2.12}$$

From (2.12) we obtain the equality

$$Cone(y + S) \cup \{0\} = \{x \mid \langle u_k, x\rangle \geq 0,\ k = 1, 2, \ldots, n\}.$$

Since $y + S \subset \text{Int}(C)$, it follows that

$$C_m^> = \{x \in \mathbf{R}^n \mid \langle u_j, x\rangle > 0,\ j = 1, 2, \ldots, m\} \subset \text{Int}(C).$$

■

Further on, we shall investigate a particular case where A contains only one point.

PROPOSITION 2.63 *Let B be a nonempty subset of $\mathbf{R}^n$, $y \notin B$, and $J = \{1, 2, \ldots, n\}$. The point y and the set B are cone-separable by a cone C such that $y \in$ Int(C) if and only if there exist linearly independent vectors $u_1, u_2, \ldots, u_n$ such that*

$$\min\{\langle u_j, y\rangle \mid j \in J\} > 0 \tag{2.13}$$

and

$$\min\{\langle u_j, y\rangle \mid j \in J\} > \sup\{\min\{\langle u_j, x\rangle \mid j \in J\} \mid x \in B\}. \tag{2.14}$$

PROOF. Assume that there exists a convex solid cone C such that

$$(y + \text{Int}(C)) \cap B = \emptyset$$

and let $y \in \text{Int}(C)$. Then there exists ε, $0 < \varepsilon < 1$, such that $(1-\varepsilon)y \in \text{Int}(C)$. Let $z = (1 - \varepsilon)y \in \text{Int}(C)$. By Proposition 2.62 there exist linearly independent vectors $u_1, u_2, \ldots, u_n$ such that $\langle u_j, z\rangle = 1$ for all $j \in J$. Moreover,

$$C_n^> = \{x \in \mathbf{R}^n \mid \langle u_j,\ x\rangle > 0,\ j \in J\} \subset \text{Int}(C).$$

If $x \in z + C_n^>$, then $\langle u_j, x\rangle > 1$ for all $j \in J$. Let $x \in B$. Then $x \notin z + \mathrm{Int}(C)$, therefore $x \notin z + C_n^>$. Consequently, there exists an index i such that $\langle u_i, x\rangle \leq 1$. Thus we have

$$\begin{aligned}\min\{\langle u_j, x\rangle \mid j \in J\} \leq 1 &= \min\{\langle u_j, z\rangle \mid j \in J\} \\ &= (1-\varepsilon)\min\{\langle u_j, y\rangle \mid j \in J\}.\end{aligned}$$

It follows that $\min\{\langle u_j, y\rangle \mid j \in J\} > 0$ and

$$\begin{aligned}\sup\{\min\{\langle u_j, x\rangle \mid j \in J\} \mid x \in B\} &\leq (1-\varepsilon)\min\{\langle u_j, y\rangle \mid j \in J\} \\ &< \min\{\langle u_j, y\rangle \mid j \in J\}.\end{aligned}$$

On the other hand, assume that (2.13) and (2.14) hold for some linearly independent vectors $u_1, u_2, \ldots, u_n$. Let $\gamma > 0$ be a number with

$$\sup\{\min\{\langle u_j, x\rangle \mid j \in J\} \mid x \in B\} < \gamma < \min\{\langle u_j, y\rangle \mid j \in J\}.$$

Consider the convex cone $C_n^> = \{x \in \mathbf{R}^n \mid \langle u_j, x\rangle > 0, j \in J\}$. Since $\langle u_j, x\rangle > 0$ for all $j \in J$, it follows that $y \in C_n^>$ and $C_n^>$ is a nonempty open cone, hence $C_n^> = \mathrm{Int}(C_n^>)$. Moreover, we have

$$\begin{aligned}y + C_n^> &\subset \{z \in \mathbf{R}^n \mid \langle u_j, z\rangle > \gamma, j \in J\} \\ &= \{z \in \mathbf{R}^n \mid \min\{\langle u_j, z\rangle \mid j \in J\} > \gamma\}.\end{aligned}$$

Consequently, $(y + C_n^>) \cap B = \emptyset$. ■

It follows from Proposition 2.63 that if a set A and a point a, that does not belong to A, are cone separated, and, in addition, point a belongs to the interior of the separating cone, then A and a are strictly separated by a family of n linearly independent linear functionals.

5.5. Separation of Starshaped Sets

Now we present two results concerning separation of two starshaped sets by a finite family of linearly independent linear functionals. We start with the strict separation of a point and a starshaped set by n linearly independent linear functionals.

THEOREM 2.64 *Let B be a nonempty closed starshaped subset of $\mathbf{R}^n$, let $\theta \in B$ and $y \notin B$. Then y and B are strictly separable by a family of n linearly independent linear functionals.*

PROOF. First we prove that there exists a closed solid convex cone C and $\lambda > 0$ such that

$$y \in \mathrm{Int}(C) \text{ and } B \cap (\lambda y + C) = \emptyset. \tag{2.15}$$

Since B is a closed starshaped set, it follows that there exists $0 < \lambda < 1$ such that $x = \lambda y \notin B$. We claim that x and B can be separated by a closed convex cone C such that $x \in \text{Int}(C)$. If, in contrary, it is not true, then there exists a sequence $\{C_k\}$ of closed solid convex cones with

$$C_{k+1} \setminus \{\theta\} \subset \text{Int}(C_k),$$

$$\bigcap_{k=1}^{\infty} C_k = \{z \in \mathbf{R}^n \mid z = \lambda y,\ \lambda \geq 0\}$$

and for each $k = 1, 2, \ldots$, there exists $x_k \in C_k$ such that $y_k \in B$, where

$$y_k = x + x_k. \tag{2.16}$$

Consider two cases:

(1) The sequence $\{x_k\}$ is unbounded, i.e., $\|x_k\| \to +\infty$ for $k \to +\infty$. Multiplying both sides of (2.16) by an arbitrary positive number γ and dividing by $\|x_k\|$, we conclude by starshapedness of B that

$$\frac{\gamma y_k}{\|x_k\|} \in B,$$

for sufficiently large k, which is impossible.

(2) The sequence $\{x_k\}$ is bounded. Without loss of generality suppose that $x_k \to x$. Then there exists $\delta > 0$ such that $x = \delta y$, hence $y_k \to x + \delta y = (\lambda + \delta)y$. Since $y_k \in B$, we have $(\lambda + \delta)y \in B$. On the other hand, $(\lambda + \delta)y \notin B$ as $\lambda y \notin B$ and B is starshaped, a contradiction. Consequently, x and B can be separated by a closed convex cone C such that $x \in \text{Int}(C)$, and (2.15) holds. Proposition 2.62 guarantees that there exist n linearly independent vectors $u_1, u_2, \ldots, u_n$ such that

$$C_n^{>} = \{z \in \mathbf{R}^n \mid \langle u_j, z\rangle > 0, j \in J\} \subset \text{Int}(C)$$

and $\langle u_j, y\rangle = 1$ for all $j = 1, 2, \ldots, n$. It is easy to see that

$$\lambda y + C_n^{>} = \{z \in \mathbf{R}^n \mid \langle u_j,\, z\rangle > 0, j \in J\}.$$

Since $B \cap (\lambda y + C_n^{>}) = \emptyset$, it follows that for each $b \in B$, there exists some index i such that $\langle u_i, b\rangle \leq \lambda$. ∎

Now, we formulate a more general result concerning separation of two starshaped sets by a finite family of linearly independent linear functionals. The proof is omitted here. It can be found in [110] and [112].

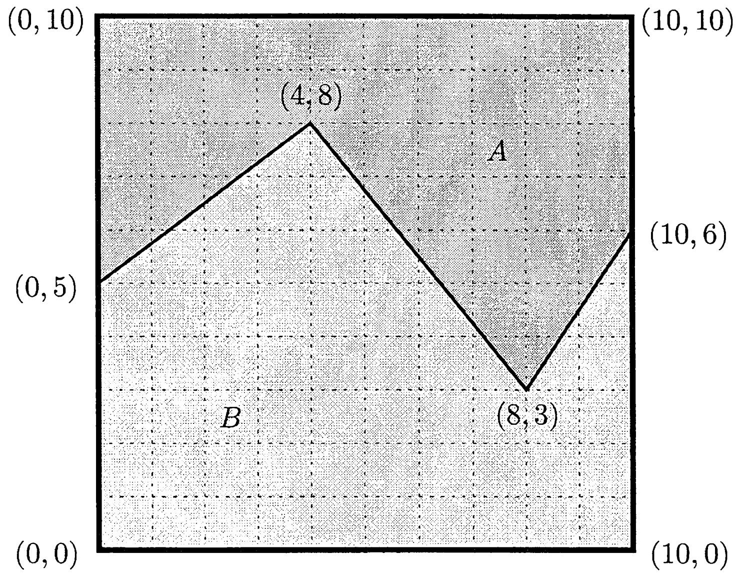

Figure 2.4.

THEOREM 2.65 *Let A and B be nonempty starshaped subsets of $\mathbf{R}^n$ such that*

(i) $Int\,(\mathrm{Ker}(A)) \neq \emptyset$,

(ii) $Int\,(A) \cap B = \emptyset$.

Then A and B are separable by a family of n linearly independent functionals.

Assumption (i) in Theorem 2.65 can be weakened, a discussion of this problem can be found in [110]. The following example, borrowed from [112], shows that separability by a finite family of linear functionals may lose its visual meaning well known for separability by a finite family of hyperplanes.

EXAMPLE 2.66 Consider a square $[0, 10] \times [0, 10]$ and a polygonal line which cuts this square into two starshaped sets A and B satisfying assumptions (i) and (ii) of Theorem 2.65, see Figure 2.4. By Theorem 2.65, A and B are separable by two linearly independent linear functionals. Let for $(x_1, x_2) \in \mathbf{R}^2$,

$$f_1\,(x_1, x_2) = -\frac{3}{23}x_1 + \frac{1}{23}x_2 \text{ and } f_2\,(x_1, x_2) = \frac{1}{3}x_1 + \frac{1}{3}x_2.$$

It can be verified that A and B are separable by the family of linearly independent linear functionals $\{f_1, f_2\}$. □

6. Generalizations of Starshaped Sets

In this section we present three generalizations of convex and starshaped sets. The first two are known from literature, namely path-connected sets and univex sets; see e.g. [3], [7], [9], [55], [74], [131]. The last and the most general one, namely, the Φ-convex sets, is new.

6.1. Path-Connected Sets

If, in the definition of a starshaped set, we allow that each point from the kernel may be connected with every other point by a path, i.e., by a continuous arc that belongs to this set, then we obtain a more general class of sets; see [3].

DEFINITION 2.67 *Let X be a nonempty subset of $\mathbf{R}^n$ and $x, y \in \mathbf{R}^n$. Let $\varphi : [0,1] \to \mathbf{R}^n$ be a continuous mapping such that $\varphi(0) = x$ and $\varphi(1) = y$. The set*

$$\mathbf{P}(x,y) = \{z \in \mathbf{R}^n \mid z = \varphi(\lambda),\ \lambda \in [0,1]\}$$

is called a path connecting x and y.

Notice that for given $x, y \in \mathbf{R}^n$, the line segment $\mathbf{I}(x,y)$ joining x and y is a path connecting x and y, i.e.,

$$\mathbf{I}(x,y) = \{z \in \mathbf{R}^n \mid z = x + \lambda(y-x),\ \lambda \in [0,1]\} = \mathbf{P}(x,y).$$

Here, $\varphi(\lambda) = x + \lambda(y - x)$.

DEFINITION 2.68 *Let X be a nonempty subset of $\mathbf{R}^n$. The set X is said to be* path-connected *if, for every $x, y \in X$, $x \neq y$, there exists a path $\mathbf{P}(x,y)$ connecting x and y such that $\mathbf{P}(x,y) \subset X$.*

If X is a nonempty starshaped set, then X is path-connected. Indeed, any two distinct points x and y from X can be connected by a path consisting of two line segments connecting x and y through a point in $\mathrm{Ker}(X)$. Moreover, in 1-dimensional space $\mathbf{R}$, convex sets, starshaped sets and path-connected sets coincide. It is obvious that a path connected set is connected, see Chapter 1. The converse statement is, however, not true as the following example demonstrates.

EXAMPLE 2.69 Let $X_1 = \{(x,y) \in \mathbf{R}^2 \mid x = 0, y \in [-1,1]\}$, $X_2 = \{(x,y) \in \mathbf{R}^2 \mid y = \sin\frac{1}{x},\ x \in (0,1]\}$, $X = X_1 \cup X_2$. It can be shown that X is connected, but it is not path-connected. □

The above example illustrates that only some "pathological" connected sets are not path-connected and, therefore, the class of path-connected sets is fairly large.

6.2. Invex and Univex Sets

In the previous subsection the concept of path-connected set was based on the existence of a certain object - a path joining any two given points of the set under consideration. To generalize convex sets in a different way, we first observe that the line segment connecting given points x and y is the set of all point $x + \lambda(y - x)$ with $\lambda \in [0, 1]$. Obviously we can consider the expression $x + \lambda(y - x)$ as a special case of the expression $x + \lambda\eta(x, y)$ where η is a mapping of the Cartesian product $X \times X$ into X. This leads to the following definition.

DEFINITION 2.70 *Let X be a nonempty subset of $\mathbf{R}^n$, let η be a mapping, $\eta : X \times X \to \mathbf{R}^n$. The set X is said to be* invex with respect to η *if for each $x, y \in X$ and $\lambda \in [0, 1]$ we have*

$$x + \lambda\eta(x, y) \in X. \tag{2.17}$$

Clearly, every convex set is invex with respect to $\eta(x, y) = y - x$. Notice that each nonempty subset X of $\mathbf{R}^n$ is invex with respect to η that maps every point of $X \times X$ to the zero vector of $\mathbf{R}^n$.

A path-connected set is not necessarily invex with respect to some given η. For example, each circle in $\mathbf{R}^2$ is path-connected; it is, however, not invex with respect to $\eta(x, y) = y - x$, because it is not convex. The next proposition gives some sufficient conditions for an invex set to be path-connected.

PROPOSITION 2.71 *Let X be a nonempty subset of $\mathbf{R}^n$, let X be invex with respect to $\eta : X \times X \to \mathbf{R}^n$. Let for every $x \in X$ the function $F_x : X \to \mathbf{R}^n$ defined for $y \in X$ by $F_x(y) = x + \eta(x, y)$, satisfy*

$$X \subset \mathrm{Ran}(F_x). \tag{2.18}$$

Then X is path-connected.

PROOF. Let $x, y \in X$. By (2.18) there exists $w \in X$ such that

$$F_x(w) = x + \eta(x, w) = y. \tag{2.19}$$

Let $\lambda \in [0, 1]$ and let

$$\varphi(\lambda) = x + \lambda\eta(x, w). \tag{2.20}$$

Then by (2.20) and (2.19) we obtain $\varphi(0) = x$ and $\varphi(1) = y$. Moreover, by (2.20) φ is continuous on $[0, 1]$. Since X is invex, then by (2.17) and (2.20) we get $\varphi(\lambda) \in X$ for all $\lambda \in [0, 1]$. Consequently, by Definition 2.67, X is path-connected. ∎

The following definition introduced in [10] gives a further extension of the concept of convex set. This definition will be useful in the next chapter for investigating generalized concave functions.

DEFINITION 2.72 *Let X be a nonempty subset of $\mathbf{R}^n$ and let η be a mapping that maps $X \times X$ into $\mathbf{R}^n$. Moreover, let $b : X \times X \times [0,1] \to [0,+\infty)$ and $\xi : \mathbf{R} \to \mathbf{R}$ be given functions. A nonempty subset S of $\mathbf{R}^n \times \mathbf{R}$ is said to be* univex with respect to η, b and ξ *if for each $(x,\alpha), (y,\beta) \in S$ and $0 \le \lambda \le 1$ we have*

$$(x + \lambda\eta(x,y), \alpha + \lambda b(x,y,\lambda)\xi(\beta - \alpha)) \in S.$$

Apparently, every nonempty convex subset of $\mathbf{R}^n \times \mathbf{R}$ is univex with respect to $\eta(u,v) = v - u$, $b(u,v,\lambda) = 1$, $\xi(t) = t$.

A univex set with respect to η, b and ξ, where b is independent on λ, is invex with respect to $\hat{\eta} = (\eta, b\xi)$.

Evidently, a path-connected set is not necessarily univex with respect to some given η, b and ξ. In view of Proposition 2.71, it is possible to formulate some sufficient conditions for a univex set to be path-connected. This is, however, left to the reader.

6.3. Φ-Convex Sets

On the one hand, the path-connectedness of a set requires that, for arbitrary two points of the set, there is at least one path, a curve, connecting these points and belonging to the set. However the "shape" of this curve is not precisely specified. On the other hand, the invexity of a set specifies a particular "shape" of the curve by function η. Both approaches can be unified by the following concept.

DEFINITION 2.73 *Let X be a nonempty subset of $\mathbf{R}^n$, let Φ be a set of mappings $\varphi : X \times X \times [0,1] \to \mathbf{R}^n$. The set X is said to be* Φ-convex *if for every $x, y \in X$ there exists $\varphi \in \Phi$ such that for each $\lambda \in [0,1]$ we have $\varphi(x,y,\lambda) \in X$.*

Clearly, if X is an invex set with respect to η, then X is Φ-convex, where Φ consists of a single function defined by $\varphi(x,y,\lambda) = x + \lambda\eta(x,y)$ for each $x, y \in X$ and $\lambda \in [0,1]$.

We also claim that each path-connected set X is Φ-convex. Indeed, let X be path-connected. Then for each $x, y \in X$ there exists a continuous function $\varphi'_{x,y} : [0,1] \to \mathbf{R}^n$ such that $\varphi'_{x,y}(0) = x$, $\varphi'_{x,y}(1) = y$ and $\varphi'_{x,y}(\lambda) \in X$ for all $\lambda \in [0,1]$. Let Φ consists of all functions $\varphi_{x,y} : X \times X \times [0,1] \to \mathbf{R}^n$ defined by $\varphi_{x,y}(u,v,\lambda) = \varphi'_{x,y}(\lambda)$ for all $u, v \in X$, $\lambda \in [0,1]$. It is obvious that X is Φ-convex.

It was mentioned before that both convex and starshaped sets are path-connected, hence they are also Φ-convex for particular classes of functions Φ.

Chapter 3

GENERALIZED CONCAVE FUNCTIONS

1. Concave and Quasiconcave Functions

The notion of concavity of real-valued functions of real variables and its various generalizations have found many applications in economics and engineering. We refer to [3] for a detailed treatment of concavity and some of its generalizations.

The concept of concavity, convexity, quasiconcavity, quasiconvexity and quasimonotonicity of a function $f : \mathbf{R}^n \to \mathbf{R}$ can be introduced in several ways. The following definitions will be most suitable for our purpose.

DEFINITION 3.1 *Let X be a nonempty subset of $\mathbf{R}^n$. A function $f : X \to \mathbf{R}$ is called*

(i) concave on X (CA) *if*

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y), \tag{3.1}$$

for every $x, y \in X$ and every $\lambda \in (0,1)$ with $\lambda x + (1-\lambda)y \in X$;

(ii) strictly concave on X *if*

$$f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y), \tag{3.2}$$

for every $x, y \in X$, $x \neq y$ and every $\lambda \in (0,1)$ with $\lambda x + (1-\lambda)y \in X$;

(iii) semistrictly concave on X *if f is concave on X and (3.2) holds for every $x, y \in X$ and every $\lambda \in (0,1)$ with $\lambda x + (1-\lambda)y \in X$ such that $f(x) \neq f(y)$;*

(iv) convex on X (CV) *if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

for every $x, y \in X$ *and every* $\lambda \in (0,1)$ *with* $\lambda x + (1-\lambda)y \in X$*;*

(v) strictly convex on X *if*

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y), \qquad (3.3)$$

for every $x, y \in X, x \neq y$ *and every* $\lambda \in (0,1)$ *with* $\lambda x + (1-\lambda)y \in X$*;*

(vi) semistrictly convex on X *if* f *is convex on* X *and (3.3) holds for every* $x, y \in X$ *and every* $\lambda \in (0,1)$ *with* $\lambda x + (1-\lambda)y \in X$ *such that* $f(x) \neq f(y)$*;*

DEFINITION 3.2 *Let* X *be a nonempty subset of* $\mathbf{R}^n$. *A function* $f : X \to \mathbf{R}$ *is called*

(i) quasiconcave on X (QCA) *if*

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\},$$

for every $x, y \in X$ *and every* $\lambda \in (0,1)$ *with* $\lambda x + (1-\lambda)y \in X$*;*

(ii) strictly quasiconcave on X *if*

$$f(\lambda x + (1-\lambda)y) > \min\{f(x), f(y)\}, \qquad (3.4)$$

for every $x, y \in X, x \neq y$ *and every* $\lambda \in (0,1)$ *with* $\lambda x + (1-\lambda)y \in X$*;*

(iii) semistrictly quasiconcave on X *if* f *is quasiconcave on* X *and (3.4) holds for every* $x, y \in X$ *and every* $\lambda \in (0,1)$ *with* $\lambda x + (1-\lambda)y \in X$ *such that* $f(x) \neq f(y)$*;*

(iv) quasiconvex on X (QCV) *if*

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \qquad (3.5)$$

for every $x, y \in X$ *and every* $\lambda \in (0,1)$ *with* $\lambda x + (1-\lambda)y \in X$*;*

(v) strictly quasiconvex on X *if*

$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}, \qquad (3.6)$$

for every $x, y \in X, x \neq y$ *and every* $\lambda \in (0,1)$ *with* $\lambda x + (1-\lambda)y \in X$*;*

(vi) semistrictly quasiconvex on X *if* f *is quasiconvex on* X *and (3.6) holds for every* $x, y \in X$ *and every* $\lambda \in (0,1)$ *with* $\lambda x + (1-\lambda)y \in X$ *such that* $f(x) \neq f(y)$*;*

(vii) quasimonotone on X (QM) *if it is both quasiconcave and quasiconvex on* X*;*

(viii) strictly quasimonotone on X *if it is both strictly quasiconcave and strictly quasiconvex on* X.

Notice that in Definitions 3.1 and 3.2 the set X is not required to be convex. If in the above definitions the set X is convex, then we obtain the usual definition of (strictly) quasiconcave and (strictly) quasiconvex functions. Observe that if a function is (strictly) concave and (strictly) convex on X, then it is (strictly) quasiconcave and (strictly) quasiconvex on X, respectively, but not vice-versa.

In Definitions 3.1 and 3.2 we introduced concepts of semistrictly CA functions and semistrictly QCA functions, respectively. The former (the latter) is stronger than the concept of a CA function (QCA function), and weaker than the concept of a strictly CA function (strictly QCA function). An example of a semistrictly CA function that is not strictly CA is depicted in Figure 3.1 (a), whereas Figure 3.1 (b) shows a semistrictly QCA function that is not strictly quasiconcave. Notice that both functions are constant on the interval $[0, 1]$.

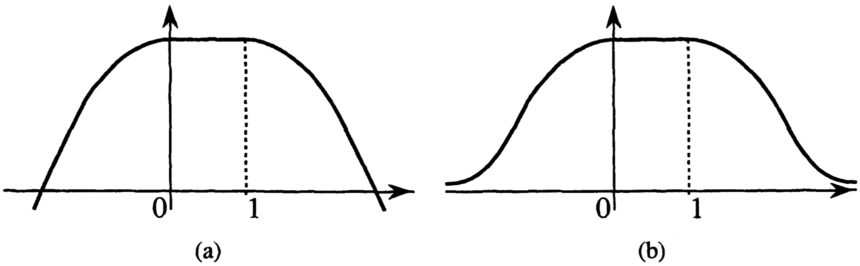

Figure 3.1.

Apart from the concavity, quasiconcavity, and their variants, the following closely related concepts will be useful later.

Let f be a real-valued function defined on a subset X of $\mathbf{R}^n$. Moreover, let $\mathbb{L}_X(x, y)$ and $f_{x,y}$ be defined for $x, y \in X$, $x \neq y$, by

$$\mathbb{L}_X(x, y) = \{t \in \mathbf{R} \mid x + t(y - x) \in X\},$$

and

$$f_{x,y}(t) = f(x + t(y - x))$$

for every $t \in \mathbb{L}_X(x, y)$. The function $f_{x,y}$ is a restriction from $\mathbf{R}$ to $\mathbb{L}_X(x, y)$ of the composite function $f \circ \varphi$ where $\varphi : \mathbf{R} \to \mathbf{R}^n$ is defined by $\varphi(t) = x + t(x - y)$. We introduce the following definition.

DEFINITION 3.3 *Let X be a subset of $\mathbf{R}^n$, $y \in X$. A function $f : X \to \mathbf{R}$ is called*

(i) (strictly) concave on X from y *if $f_{x,y}$ is (strictly) concave on $\mathbb{L}_X(x,y)$ for every $x \in X, x \neq y$;*

(ii) (strictly) convex on X from y *if $f_{x,y}$ is (strictly) convex on $\mathbb{L}_X(x,y)$ for every $x \in X, x \neq y$;*

(iii) (semistrictly, strictly) quasiconcave on X from y *if $f_{x,y}$ is (semistrictly, strictly) quasiconcave on $\mathbb{L}_X(x,y)$ for every $x \in X, x \neq y$;*

(iv) (semistrictly, strictly) quasiconvex on X from y *if $f_{x,y}$ is (semistrictly, strictly) quasiconvex on $\mathbb{L}_X(x,y)$ for every $x \in X, x \neq y$;*

(v) (strictly) quasimonotone on X from y *if it is both (strictly) quasiconcave and (strictly) quasiconvex on X from y.*

Observe that a real-valued function f, defined on a set $X \subset \mathbf{R}^n$, is concave (convex) on X, if and only if, for every $y \in X$, f is concave (convex) on X from y, respectively. The same is true for (semistrictly, strictly) QCA and (semistrictly, strictly) QCV functions on X.

It is worth noting that f is (strictly) (quasi)concave on X (from y) if and only if $-f$ is (strictly) (quasi)convex on X (from y).

Obviously, the class of quasiconcave functions on $X \subset \mathbf{R}$ coincides with the class of functions being quasiconcave on X from y for some $y \in X$. The same assertion is, however, not true for $\mathbf{R}^n$ with $n > 1$, as is clear from the following example.

EXAMPLE 3.4 Let $A = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1^2 + x_2^2 = 1, x_1 \neq 1\}$ and let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined as follows:

$$f(x_1, x_2) = \begin{cases} 1 & \text{for } (x_1, x_2) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

It can easily be verified that f is quasiconcave on $\mathbf{R}^2$ from $y = (1, 0)$. Clearly f is not quasiconcave on A. □

2. Starshaped Functions

Starshaped functions are more general than quasiconcave (quasiconvex) functions. We begin this section with two propositions, which give a well known characterization of concave and convex and quasiconcave (quasiconvex) functions. The corresponding proofs are simple and can be found in [3] or [67].

PROPOSITION 3.5 *Let X be a convex subset of $\mathbf{R}^n$. A function $f : X \to \mathbf{R}$ is concave on X if and only if its hypograph is a convex subset of $\mathbf{R}^{n+1}$. Likewise, f is convex on X if and only if its epigraph is a convex subset of $\mathbf{R}^{n+1}$.*

PROPOSITION 3.6 *Let X be a convex subset of $\mathbf{R}^n$. A function $f : X \to \mathbf{R}$ is quasiconcave on X if and only if all its upper-level sets are convex subsets of $\mathbf{R}^n$. Likewise, f is quasiconvex on X if and only if all its lower-level sets are convex subsets of $\mathbf{R}^n$.*

Propositions 3.5 and 3.6 suggest two ways of generalization of the concave (convex) functions. Replacing the convexity of hypograph Hyp(f) (convexity of epigraph Epi(f)) in Proposition 3.5 by generalized convex sets, we obtain a generalized concave (convex) function. This idea will be applied in the following section where univex functions will be defined. Similarly, replacing all convex upper-level sets $U(f, \alpha)$ (convex lower-level sets $L(f, \alpha)$) in Proposition 3.6 by generalized convex sets, we obtain generalized quasiconcave (quasiconvex) functions. The third way of generalizing concave and convex functions is to extend formula (3.1) and (3.2), respectively. This way will also be followed in the sequel.

We begin with a generalization of the concept of quasiconcave functions from Definition 3.2 and Proposition 3.6 by replacing convexity of upper-level sets with starshapedness. The starshaped functions are of a particular interest, since, later on, in Chapter 4, there will be another new class of generalized concave functions, called T-quasiconcave functions, which will be extensively used in fuzzy optimization in Part II.

DEFINITION 3.7 *Let X be a starshaped subset of $\mathbf{R}^n$. A function $f : X \to \mathbf{R}$ is called*

(i) upper-starshaped on X (US) *if its upper-level sets $U(f, \alpha)$ are starshaped subsets of $\mathbf{R}^n$ for all $\alpha \in \mathbf{R}$;*

(ii) lower-starshaped on X (LS) *if its lower-level sets $L(f, \alpha)$ are starshaped subsets of $\mathbf{R}^n$ for all $\alpha \in \mathbf{R}$;*

(iii) monotone-starshaped on X (MS) *if it is both lower-starshaped and upper-starshaped on X.*

It is obvious that if a function $f : X \to \mathbf{R}^n$ is upper-starshaped on X, then the function $-f$ is lower-starshaped on X, and vice-versa.

From the fact that each convex set is starshaped it follows that each quasiconcave (quasiconvex) function is upper-starshaped (lower-starshaped). Moreover, each quasimonotone function is monotone-starshaped. Evidently, the classes of quasiconcave (quasiconvex) functions and upper-starshaped (lower-starshaped) functions coincide on $\mathbf{R}$.

In the following examples we demonstrate that on $\mathbf{R}^2$ the situation is not so simple as in $\mathbf{R}$. We demonstrate that an upper-starshaped function on a convex set is not necessarily quasiconcave.

EXAMPLE 3.8 Let $X_7 = \{(x, y) \in \mathbf{R}^2 \mid x > 0,\ y > 0\}$ and let f be defined for $(x, y) \in X_7$ as follows: $f(x, y) = \max\{0, x(y-1), (x-1)y\}$, see Figure 3.2. If $\alpha > 0$, then

$$\begin{aligned} &U(f, \alpha) \\ &\quad = \{(x, y) \in X_7 \mid x(y-1) \geq \alpha\} \cup \{(x, y) \in X_7 \mid (x-1)y \geq \alpha\} \\ &\quad = \left\{(x, y) \in X_7 \;\middle|\; y \geq \tfrac{\alpha}{x} + 1\right\} \cup \left\{(x, y) \in X_7 \;\middle|\; y \geq \tfrac{\alpha}{x-1},\ x > 1\right\}, \end{aligned}$$

see Figure 3.3, where $\alpha = 2$. Evidently, if $\alpha \leq 0$, then $U(f, \alpha) = X_7$. We can see that not every upper level set is convex. However, all of them are starshaped. It follows that f is upper-starshaped on X_7. Moreover, let $x_\alpha = y_\alpha = (1 + \sqrt{1 + 4\alpha})/2$, $\alpha > 0$. It can be also shown that the point (x_α, y_α) belongs to the kernel $\mathrm{Ker}(U(f, \alpha))$. □

EXAMPLE 3.9 Let $X_8 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$ and let a function $f : \mathbf{R}^2 \to \mathbf{R}$ be defined on $\mathbf{R}^2$ as follows:

$$f(x, y) = \begin{cases} 1 & \text{if } x = r\cos\varphi,\ y = r\sin\varphi, \\ & \varphi \text{ is a rational number from } [0, 2\pi), \\ 0 & \text{otherwise.} \end{cases}$$

For $\alpha \leq 0$ we obtain $U(f, \alpha) = \mathbf{R}^2$, for $0 < \alpha \leq 1$ we have $U(f, \alpha) = X_4$, where X_4 is a starshaped set defined in Example 2.9. For $1 < \alpha$, apparently $U(f, \alpha) = \emptyset$. Hence, by Definition 3.7, f is upper-starshaped on $\mathbf{R}^2$ (and also on X_8, and on X_4). Notice that f is discontinuous at each point $(x, y) \in X_8$. □

EXAMPLE 3.10 Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined as follows:

$$f(x, y) = \begin{cases} \max\{0, xy\} & \text{for } x > 0, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $\alpha < 0$ we have $U(f, \alpha) = \mathbf{R}^2$ and $L(f, \alpha) = \emptyset$. For $\alpha = 0$ we have $U(f, \alpha) = L(f, \alpha) = \mathbf{R}^2$. For $0 < \alpha$ we get

$$\begin{aligned} U(f, \alpha) &= \{(x, y) \in \mathbf{R}^2 \mid y \geq \frac{\alpha}{x},\ x > 0\}, \\ L(f, \alpha) &= Cl\left(\mathbf{R}^2 \setminus U(f, \alpha)\right). \end{aligned}$$

It follows that, for each real number α, both $U(f, \alpha)$ and $L(f, \alpha)$ are starshaped sets. Hence f is both upper-starshaped and lower-starshaped, i.e., it is

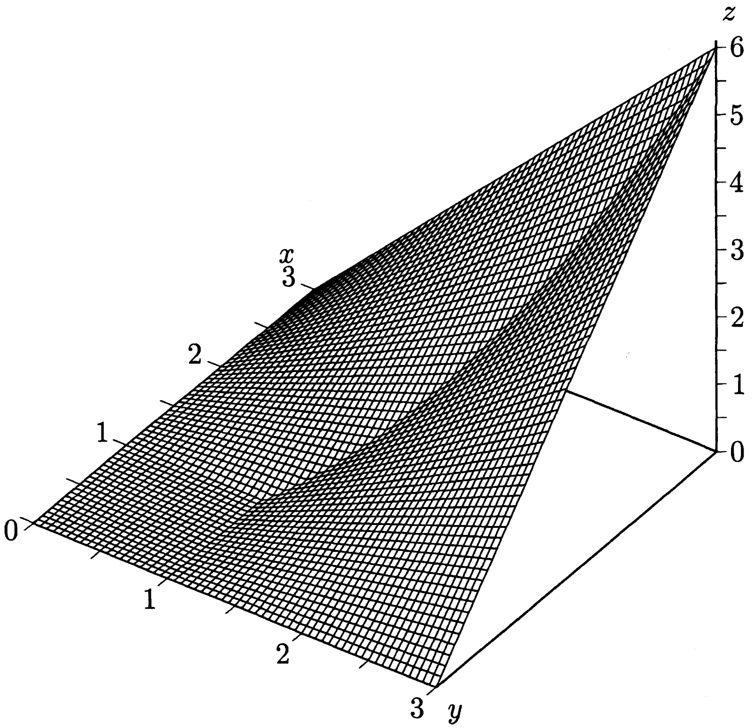

Figure 3.2.

a monotone-starshaped function on $\mathbf{R}^2$. Notice also that f is continuous on $\mathbf{R}^2$ and that it is not quasimonotone on $\mathbf{R}^2$. □

The following theorems give characterizations of a class of upper-starshaped (lower-starshaped, monotone-starshaped) functions by means of quasiconcavity (quasiconvexity, quasimonotonicity) from special points. The following notation will be used. We denote

$$\begin{aligned} I^f &= \{\alpha \in \mathbf{R} \mid U(f,\alpha) \neq \emptyset\}, \\ I_f &= \{\alpha \in \mathbf{R} \mid L(f,\alpha) \neq \emptyset\}. \end{aligned}$$

THEOREM 3.11 *Let X be a starshaped subset of $\mathbf{R}^n$, let $f : X \to \mathbf{R}$ be a real-valued function on X. If*

$$\bar{x} \in \bigcap_{\alpha \in I^f} \operatorname{Ker}(U(f,\alpha)), \tag{3.7}$$

then f is quasiconcave on X from $\bar{x}$.

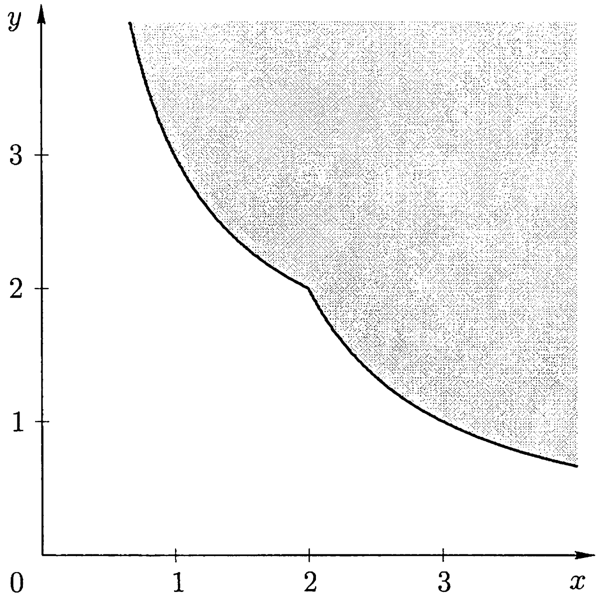

Figure 3.3.

PROOF. By (3.7) f is upper-starshaped on X. To prove the theorem, by Definition 3.3, it is sufficient to show that if $\bar{x}$ satisfies (3.7) and x is a point of X different from $\bar{x}$, then $f_{x,\bar{x}}$ is quasiconcave on $[0,1]$. Let $x \in X$, $y = x + \lambda(\bar{x} - x)$ for arbitrary $\lambda \in (0,1)$, set $\beta = f(x)$, i.e., $\beta = f_{x,\bar{x}}(0)$. Then $x \in U(f,\beta)$, $\bar{x} \in \operatorname{Ker}(U(f,\beta))$ and since f is upper-starshaped on X, it follows that the upper-level set $U(f,\beta)$ is starshaped, consequently, $y = x + \lambda(\bar{x} - x) \in U(f,\beta)$. Hence,

$$f(y) = f_{x,\bar{x}}(\lambda) \geq \beta = f_{x,\bar{x}}(0).$$

From the last inequality we obtain that $f_{x,\bar{x}}$ is nondecreasing on $[0,1]$, that is, f is quasiconcave on X from $\bar{x}$. ■

Notice that if condition (3.7) is satisfied, then f is upper-starshaped on X. Example 3.4 demonstrates that an opposite statement to the above theorem is not true. Particularly, f is not upper-starshaped on A, since the upper level set $U(f,1) = A$ and A is not starshaped.

The analogous result to Theorem 3.11 can be formulated for lower-starshaped and quasiconvex functions. The proof of the preceding theorem can be easily modified for this case.

THEOREM 3.12 *Let X be a starshaped subset of $\mathbf{R}^n$, let $f : X \to \mathbf{R}$ be a real-valued function on X. If*

$$\hat{x} \in \bigcap_{\alpha \in I_f} \mathrm{Ker}\,(L(f, \alpha)). \tag{3.8}$$

then f is quasiconvex on X from $\hat{x}$.

Notice that condition (3.7), (3.8), i.e.,

$$\bigcap_{\alpha \in I^f} \mathrm{Ker}\,(U(f, \alpha)) \neq \emptyset, \quad \bigcap_{\alpha \in I_f} \mathrm{Ker}\,(L(f, \alpha)) \neq \emptyset,$$

are not satisfied in general, e.g. in Example 3.8, we obtain

$$\bigcap_{\alpha \in I^f} \mathrm{Ker}\,(U(f, \alpha)) = \emptyset,$$

see also the following Example 3.13. We return to this problem in the next chapter, where we deal with Similarly, we deal with necessary and sufficient conditions for validity of (3.7) and (3.8). To throw more light on this problem, consider another example.

EXAMPLE 3.13 Let $B \in \mathbf{R}^2$ is one of the end points of the "moon-shaped" set X_2 from Example 2.7, see Figure 2.1 (b). Let $f : \mathbf{R}^2 \to [0, 1]$ be defined as follows:

$$f(x_1, x_2) = \begin{cases} 1 & \text{for } (x_1, x_2) = B, \\ \frac{1}{2} & \text{for } (x_1, x_2) \in X_2 \setminus \{B\}, \\ 0 & \text{otherwise.} \end{cases}$$

As it is easily seen, the function f has the following properties:

- B is a strict global maximizer of f on $\mathbf{R}^2$,
- f is upper semicontinuous on $\mathbf{R}^2$,
- f is upper-starshaped on $\mathbf{R}^2$,
- $\bigcap_{\alpha \in I_f} \mathrm{Ker}\,(U(f, \alpha)) = \emptyset$.

□

It is only a matter of routine to modify function f in Example 3.13 in such a way that the previous properties remain valid and f is continuously differentiable. Such an example demonstrates that sufficient conditions for the validity of (3.7) should be more delicate than those mentioned in Example 3.13. This problem will be investigated by the use of triangular norms in Chapter 4.

We finish this section with a theorem characterizing monotone-starshaped functions, i.e., the functions being both upper- and lower-starshaped. Apparently, the theorem can be derived from the preceding Theorems 3.11 and 3.12 and, therefore, its proof is omitted.

THEOREM 3.14 *Let X be a starshaped subset of $\mathbf{R}^n$, let $f : X \to \mathbf{R}$ be a real-valued function on X. If*

$$\bar{x} \in \bigcap_{\alpha \in I^f} \mathrm{Ker}\,(U(f,\alpha)) \cap \bigcap_{\alpha \in I_f} \mathrm{Ker}(L(f,\alpha)),$$

then f is quasimonotone on X from $\bar{x}$.

3. Further Generalizations of Concave Functions

3.1. Quasiconnected Functions

In this subsection we further extend the notion of starshaped functions presented in the preceding section. For this purpose we utilize the concept of path-connected sets from Chapter 2; see also [3].

Recall that if $x, y \in \mathbf{R}^n$ and $\varphi : [0,1] \to \mathbf{R}^n$ is a continuous mapping such that $\varphi(0) = x$ and $\varphi(1) = y$, then the set

$$\mathbf{P}(x,y) = \{z \in \mathbf{R}^n \mid z = \varphi(\lambda),\ \lambda \in [0,1]\}$$

is called a path connecting x and y. Also recall that a subset of $\mathbf{R}^n$ is path-connected if, with every two of its points, there exists a path connecting these points and belonging to the set.

DEFINITION 3.15 *Let X be a nonempty path-connected subset of $\mathbf{R}^n$. A function $f : X \to \mathbf{R}$ is called*

(i) upper-quasiconnected on X (UQCN) *if for every $x, y \in X$, $x \neq y$ there exists a path*

$$\mathbf{P}(x,y) = \{z \in \mathbf{R}^n \mid z = \varphi(\lambda),\ \lambda \in [0,1]\} \tag{3.9}$$

connecting x and y such that $\mathbf{P}(x,y) \subset X$ and for every $\lambda \in (0,1)$

$$f(\varphi(\lambda)) \geq \min\{f(x), f(y)\}; \tag{3.10}$$

(ii) strictly upper-quasiconnected on X *if for every $x, y \in X$, $x \neq y$ there exists a path $\mathbf{P}(x,y) \subset X$, defined by (3.9), connecting x and y such that for every $\lambda \in (0,1)$*

$$f(\varphi(\lambda)) > \min\{f(x), f(y)\}; \tag{3.11}$$

(iii) semistrictly upper-quasiconnected on X *if for every* $x, y \in X$, $x \neq y$, *there exists a path* $\mathbf{P}(x, y)$, *defined by (3.9), connecting* x *and* y *such that (3.10) holds for every* $\lambda \in (0, 1)$, *and if* $f(x) \neq f(y)$, *then (3.11) holds for every* $\lambda \in (0, 1)$*;*

(iv) lower-quasiconnected on X (LQCN) *if for every* $x, y \in X$, $x \neq y$ *there exists a path* $\mathbf{P}(x, y) \subset X$, *defined by (3.9), connecting* x *and* y *such that for every* $\lambda \in (0, 1)$

$$f(\varphi(\lambda)) \leq \max\{f(x), f(y)\}; \tag{3.12}$$

(v) strictly lower-quasiconnected on X *if for every* $x, y \in X$, $x \neq y$ *there exists a path* $\mathbf{P}(x, y) \subset X$, *defined by (3.9), connecting* x *and* y *such that for every* $\lambda \in (0, 1)$

$$f(\varphi(\lambda)) < \max\{f(x), f(y)\}, \tag{3.13}$$

(vi) semistrictly lower-quasiconnected on X *if for every* $x, y \in X$, $x \neq y$, *there exists a path* $\mathbf{P}(x, y)$, *defined by (3.9), connecting* x *and* y *such that (3.12) holds for every* $\lambda \in (0, 1)$, *and if* $f(x) \neq f(y)$, *then (3.13) holds for every* $\lambda \in (0, 1)$*;*

(vii) monotone-quasiconnected on X (MQCN) *if it is both upper-quasiconnected and lower-quasiconnected on* X*;*

(viii) strictly monotone-quasiconnected on X *if it is both strictly upper-quasiconnected and strictly lower-quasiconnected on* X.

Notice that if, for $x, y \in X$ and $\lambda \in (0, 1)$, $\varphi(\lambda) = x + \lambda(y - x)$, then we obtain the definition of (strictly, semistrictly) quasiconcave function. From this observation we conclude that each (strictly, semistrictly) quasiconcave function is (strictly, semistrictly) upper-quasiconnected. Similarly, each (strictly, semistrictly) quasiconvex function is (strictly, semistrictly) lower-quasiconnected.

The following proposition gives a characterization of quasiconnected functions by upper and lower level sets.

PROPOSITION 3.16 *Let* X *be a nonempty path-connected subset of* $\mathbf{R}^n$. *A function* $f : X \to \mathbf{R}$ *is UQCN on* X *if and only if its upper-level sets* $U(f, \alpha)$ *for all* $\alpha \in \mathbf{R}$ *are path-connected subsets of* $\mathbf{R}^n$. *Likewise,* f *is LQCN on* X *if and only if all its lower-level sets* $L(f, \alpha)$ *are path-connected subsets of* $\mathbf{R}^n$.

PROOF. We prove the proposition only for UQCN functions, for LQCN functions, the proof is analogous.

1. Let f be UQCN on X, $\alpha \in \mathbf{R}$, $x, y \in U(f, \alpha)$, $x \neq y$. We show that there exists a path $\mathbf{P}(x, y) \subset X$ connecting x and y such that $\mathbf{P}(x, y) \subset U(f, \alpha)$. Since (3.10) is satisfied, it follows that there exists a path

$$\mathbf{P}(x, y) = \{z \in \mathbf{R}^n \mid z = \varphi(\lambda),\, \lambda \in [0, 1]\} \subset X$$

connecting x and y such that

$$f(\varphi(\lambda)) \geq \min\{f(x), f(y)\}$$

for every $\lambda \in [0, 1]$. Since $x, y \in U(f, \alpha)$, it follows that $\min\{f(x), f(y)\} \geq \alpha$, hence $f(\varphi(\lambda)) \geq \alpha$ for every $\lambda \in [0, 1]$, i.e., $\varphi(\lambda) \in U(f, \alpha)$ for every $\lambda \in (0, 1)$. Consequently, $\mathbf{P}(x, y) \subset U(f, \alpha)$.

2. Let $x, y \in X$, $x \neq y$. Without loss of generality we assume $f(x) \geq f(y)$. Then $x, y \in U(f, f(y))$. Since $U(f, f(y))$ is path-connected, there exists a path

$$\mathbf{P}(x, y) = \{z \in \mathbf{R}^n \mid z = \varphi(\lambda),\, \lambda \in [0, 1]\} \subset U(f, f(y))$$

connecting x and y. Therefore

$$f(\varphi(\lambda)) \geq f(y) = \min\{f(x), f(y)\}$$

for every $\lambda \in (0, 1)$, and we conclude that f is UQCN on X. ■

Further on, we shall investigate some properties of local and global extrema of UQCN functions.

THEOREM 3.17 *Let X be a nonempty path-connected subset of $\mathbf{R}^n$, let $f : \mathbf{R}^n \to \mathbf{R}$ be UQCN on X. If $\bar{x} \in X$ is a strict local maximizer of f over X, then it is a strict global maximizer of f over X.*

PROOF. Set $\alpha = f(\bar{x})$. Then the upper-level set $U(f, \alpha)$ is path-connected. Since $\bar{x} \in X$ is a strict local maximizer, there exists an open ball B with the center at $\bar{x} \in X$ such that $f(x) < f(\bar{x})$ for all $x \in X \cap B$, $x \neq \bar{x}$.

Suppose that $\bar{x} \in X$ is not a global maximizer. Then there exists $v \in X$ such that $f(\bar{x}) < f(v)$. It follows that $v \in U(f, \alpha)$ and there exists a path $\mathbf{P}(\bar{x}, v)$ connecting $\bar{x}$ and v such that $\mathbf{P}(\bar{x}, v) \subset U(f, \alpha)$ and $z = \varphi(\bar{x}, v; \lambda) \in X \cap B$ for some sufficiently small $\lambda \in (0, 1)$. Consequently, $z \in U(f, \alpha)$ and $f(z) < f(\bar{x})$, a contradiction. Hence, $\bar{x}$ is a global maximizer. Since it is also a strict local maximizer, it must be strict global one. ■

If in Theorem 3.17 we drop the assumption of strictness of the local maximizer, then clearly the assertion is no longer valid. The semistrict quasiconcavity will, however, secure the assertion.

THEOREM 3.18 *Let X be a path-connected subset of $\mathbf{R}^n$, let $f : \mathbf{R}^n \to \mathbf{R}$ be semistrictly UQCN on X. If $\bar{x} \in X$ is a local maximizer of f over X, then it is a global maximizer of f over X.*

PROOF. Set $\alpha = f(\bar{x})$. Then the upper-level set $U(f, \alpha)$ is path-connected. Since $\bar{x} \in X$ is a local maximizer, there exists an open ball B with the center at $\bar{x} \in X$, such that $f(x) \leq f(\bar{x})$ for all $x \in X \cap B$.

Suppose on contrary that $\bar{x} \in X$ is not a global maximizer. Then there exists $v \in X$, $\bar{x} \neq v$, such that $f(\bar{x}) < f(v)$. It follows that $v \in U(f, \alpha)$ and consequently there exists a path $\mathbf{P}(\bar{x}, v)$ connecting $\bar{x}$ and v, such that $\mathbf{P}(\bar{x}, v) \subset U(f, \alpha)$, $z = \varphi(\bar{x}, v; \lambda) \in X \cap B$ for some sufficiently small $\lambda \in (0, 1)$ and $f(\varphi(\bar{x}, v; \lambda)) = f(z) > f(\bar{x})$. Since $z \in X \cap B$, we have $f(z) \leq f(\bar{x})$, a contradiction. ■

Further on, we derive some sufficient conditions for the generalized quasiconcavity of composite functions. The results we will obtain are analogous to similar results for quasiconcave functions, derived in [3].

PROPOSITION 3.19 *Let X be a nonempty path-connected subset of $\mathbf{R}^n$, let $f : X \to \mathbf{R}$ be UQCN on X, let $\psi : Y \to \mathbf{R}$ be an increasing function on $Y \subset \mathbf{R}$ with $f(X) \subset Y$. Then the composite function $\psi \circ f$ is also UQCN on X.*

PROOF. On contrary, suppose that $\psi \circ f$ is not UQCN on X. Then there exists $\alpha \in \mathbf{R}$ such that the corresponding upper-level set $U(\psi \circ f, \alpha)$ is not path-connected, i.e., there exists $x, y \in U(\psi \circ f, \alpha)$ such that x and y cannot be connected by a path belonging to $U(\psi \circ f, \alpha)$. Setting $\beta = \psi^{-1}(\alpha)$, we obtain $x, y \in U(f, \beta)$. However, by the assumption, $U(f, \beta)$ is path-connected, hence there exists a path $\mathbf{P}(x, y)$ connecting x and y such that $\mathbf{P}(x, y) \subset U(f, \beta)$, where $z = \varphi(x, y, \lambda) \in U(f, \beta)$ for all $\lambda \in (0, 1)$. In other words, $f(\varphi(x, y, \lambda)) \geq \beta$, which implies that

$$\psi(f(\varphi(x, y, \lambda))) \geq \psi(\beta) = \alpha \tag{3.14}$$

for all $\lambda \in (0, 1)$. By (3.14) we conclude, that x and y can be connected by a path $\mathbf{P}(x, y)$ belonging to $U(\psi \circ f, \alpha)$, a contradiction. ■

COROLLARY 3.20 *If f is either positive or negative UQCN function on a path-connected subset X of $\mathbf{R}^n$, then $\frac{1}{f}$ is UQCN on X.*

PROOF. We can write $\frac{1}{f(x)} = \varphi(-f(x))$ with $\varphi(y) = -\frac{1}{y}$. Since $\varphi'(y) = \frac{1}{y^2} > 0$ for $y \neq 0$, by the assumption, φ is increasing over the range of $-f$. Applying Proposition 3.19, $\frac{1}{f} = \varphi \circ f$ is UQCN on X. ■

Analogical results to Theorems 3.17 and 3.18 and Proposition 3.19 could also be derived for LQCN and similar functions and their local and global minima by using the correspondence between the functions f and $-f$. The formulation of these theorems and their proofs are left to the reader.

3.2. (Φ, Ψ)-Concave Functions

In this subsection we further extend the notion of quasiconnected functions presented in the preceding section. For this purpose we utilize the concept of Φ-convex set defined in Chapter 2.

DEFINITION 3.21 *Let X be a subset of $\mathbf{R}^n$. Let Φ be a set of mappings φ with $\varphi : X \times X \times [0,1] \to \mathbf{R}^n$, Ψ be a set of functions ψ with $\psi : X \times X \times \mathbf{R} \times \mathbf{R} \times [0,1] \to \mathbf{R}$. Moreover, let X be a Φ-convex subset of $\mathbf{R}^n$. A function $f : X \to \mathbf{R}$ is called*

(i) (Φ, Ψ)-concave on X *if, for each $x, y \in X$, there exists $\varphi \in \Phi$ such that $\varphi(x, y, \lambda) \in X$ for every $\lambda \in [0,1]$ and*

$$f(\varphi(x, y, \lambda)) \geq \inf\{\psi(x, y, f(x), f(y), \lambda) \mid \psi \in \Psi\};$$

(ii) strictly (Φ, Ψ)-concave on X *if, for each $x, y \in X$, there exists $\varphi \in \Phi$ such that $\varphi(x, y, \lambda) \in X$ for every $\lambda \in (0,1)$ and*

$$f(\varphi(x, y, \lambda)) > \inf\{\psi(x, y, f(x), f(y), \lambda) \mid \psi \in \Psi\};$$

(iii) semistrictly (Φ, Ψ)-concave on X *if, for each $x, y \in X$, there exists $\varphi \in \Phi$ such that $\varphi(x, y, \lambda) \in X$ for every $\lambda \in [0,1]$ and*

$$f(\varphi(x, y, \lambda)) \geq \inf\{\psi(x, y, f(x), f(y), \lambda) \mid \psi \in \Psi\}.$$

Moreover, if $f(x) \neq f(y)$, then, for every $\lambda \in (0,1)$,

$$f(\varphi(x, y, \lambda)) > \inf\{\psi(x, y, f(x), f(y), \lambda) \mid \psi \in \Psi\};$$

(iv) (Φ, Ψ)-convex on X *if, for each $x, y \in X$, there exists $\varphi \in \Phi$ such that $\varphi(x, y, \lambda) \in X$ for every $\lambda \in [0,1]$ and*

$$f(\varphi(x, y, \lambda)) \leq \sup\{\psi(x, y, f(x), f(y), \lambda) \mid \psi \in \Psi\};$$

(v) strictly (Φ, Ψ)-convex on X *if, for each $x, y \in X$, there exists $\varphi \in \Phi$ such that $\varphi(x, y, \lambda) \in X$ for every $\lambda \in (0,1)$ and*

$$f(\varphi(x, y, \lambda)) < \sup\{\psi(x, y, f(x), f(y), \lambda) \mid \psi \in \Psi\};$$

(vi) semistrictly (Φ, Ψ)-convex on X *if, for each* $x, y \in X$, *there exists* $\varphi \in \Phi$ *such that* $\varphi(x, y, \lambda) \in X$ *for every* $\lambda \in [0, 1]$ *and*

$$f(\varphi(x, y, \lambda)) \leq \sup\{\psi(x, y, f(x), f(y), \lambda) \mid \psi \in \Psi\}.$$

Moreover, if $f(x) \neq f(y)$, *then, for every* $\lambda \in (0, 1)$,

$$f(\varphi(x, y, \lambda)) < \sup\{\psi(x, y, f(x), f(y), \lambda) \mid \psi \in \Psi\};$$

(vii) (strictly, semistrictly) (Φ, Ψ)-monotone on X *if it is both (strictly, semistrictly)* (Φ, Ψ)*-convex on* X *and (strictly, semistrictly)* (Φ, Ψ)*-concave on* X, *respectively.*

The (Φ, Ψ)-concave functions introduced in Definition 3.21 are different from (Φ, Ψ)-concave functions defined in [82], where Φ and Ψ are considered to be functions, whereas here, Φ and Ψ are considered to be classes of functions. It is obvious that the (Φ, Ψ)-concave functions defined in [82] are particular cases of (Φ, Ψ)-concave functions introduced here.

The classes of functions introduced in Definition 3.21 are remarkable rich. Considering particular sets Φ and Ψ we can identify many well known and also some new classes of generalized concave (convex) functions. As examples, consider the following:

(i) Let X be nonempty and convex, let both Φ and Ψ consist of a single function, i.e. $\Phi = \{\varphi\}$ and $\Psi = \{\psi\}$, where

$$\varphi(x, y, \lambda) = x + \lambda(y - x) \tag{3.15}$$

and

$$\psi(x, y, \alpha, \beta, \lambda) = \alpha + \lambda(\beta - \alpha) \tag{3.16}$$

for all $x, y \in X, \alpha, \beta \in \mathbf{R}$, $\lambda \in [0, 1]$. Then, by Definition 2.73, X is Φ-convex, and by Definition 3.21 we obtain (strictly, semistrictly) concave and convex functions on X.

(ii) Let Φ be the same as in paragraph (i), let Ψ consist of two functions ψ_1, ψ_2 such that

$$\psi_1(x, y, \alpha, \beta, \lambda) = \alpha \text{ and } \psi_2(x, y, \alpha, \beta, \lambda) = \beta \tag{3.17}$$

for all $x, y \in X, \alpha, \beta \in \mathbf{R}$, $\lambda \in [0, 1]$. Then by Definition 3.21 we obtain (strictly, semistrictly) quasiconcave and quasiconvex functions on convex sets.

(iii) Let Φ consist of all continuous mappings $\varphi(x, y, \cdot) : [0, 1] \to \mathbf{R}^n$ such that $\varphi(x, y, 0) = x$ and $\varphi(x, y, 1) = y$ for every $x, y \in X$, where

$\varphi : X \times X \times [0,1] \to \mathbf{R}^n$. Let Ψ be as in (ii), i.e. $\Psi = \{\psi_1, \psi_2\}$, where ψ_1, ψ_2 are defined by (3.17). Then by Definition 3.21 we obtain (strictly, semistrictly) UQCN and LQCN functions on the path-connected set X. In particular, US and LS functions are (Φ, Ψ)-concave.

(iv) Let X be a nonempty subset of $\mathbf{R}^n$, let η, b and ξ be functions, $\eta : X \times X \to \mathbf{R}^n$, $b : X \times X \times [0,1] \to [0, +\infty)$, $\xi : \mathbf{R} \to \mathbf{R}$. Now, let X be invex with respect to η. Let both Φ and Ψ consist of a single function, i.e. $\Phi = \{\varphi\}$ and $\Psi = \{\psi\}$, where

$$\varphi(x, y, \lambda) = x + \lambda\eta(x, y) \tag{3.18}$$

and

$$\psi(x, y, \alpha, \beta, \lambda) = \alpha + \lambda b(x, y, \lambda)\xi(\beta - \alpha) \tag{3.19}$$

for all $x, y \in X, \alpha, \beta \in \mathbf{R}$, $\lambda \in [0,1]$. Then by Definition 3.21 we obtain *(strictly, semistrictly) pre-unicave and pre-univex functions with respect to* η, *b and* ξ on the univex set X, introduced in [10].

(v) Let Φ be a set of mappings φ with $\varphi : X \times X \times [0,1] \to \mathbf{R}^n$, let X be a nonempty Φ-convex subset of $\mathbf{R}^n$, and let Ψ consist of a single function ψ defined by

$$\psi(x, y, \alpha, \beta, \lambda) = \alpha + \lambda(\beta - \alpha)$$

for all $x, y \in X, \alpha, \beta \in \mathbf{R}$, $\lambda \in [0,1]$. By Definition 3.21 we obtain classes of functions called *(strictly, semistrictly) Φ-concave (Φ-convex) on the set X*. Clearly, for Φ as in (i), see (3.15), we obtain the usual concave (convex) functions.

(vi) Let Φ be a given set of functions φ with $\varphi : X \times X \times [0,1] \to \mathbf{R}^n$, let X be a nonempty Φ-convex subset of $\mathbf{R}^n$ and let Ψ consist of two functions ψ_1, ψ_2 such that

$$\psi_1(x, y, \alpha, \beta, \lambda) = \alpha \text{ and } \psi_2(x, y, \alpha, \beta, \lambda) = \beta$$

for all $x, y \in X, \alpha, \beta \in \mathbf{R}$, $\lambda \in [0,1]$. By Definition 3.21 we obtain classes of functions called *(strictly, semistrictly) Φ-quasiconcave (Φ-quasiconvex) on the set X*. Note that in our new terminology an upper-quasiconnected function f is also Φ-quasiconcave, where Φ is specified in (iii). Again, for Φ as in (3.15) we obtain the usual quasiconcave (quasiconvex) functions.

The following proposition gives a characterization of pre-unicave (pre-univex) functions by their hypographs (epigraphs).

PROPOSITION 3.22 *Let X be a nonempty subset of* $\mathbf{R}^n$, *let* η, *b and* ψ *be functions,* $\eta : X \times X \to \mathbf{R}^n$, $b : X \times X \times [0,1] \to [0, +\infty)$, $\xi : \mathbf{R} \to \mathbf{R}$.

Suppose that X is invex with respect to η. Let f be a function, $f : X \to \mathbf{R}$. If the hypograph of f, (epigraph of f), is a univex set with respect to η, b and ξ, then f is pre-unicave (pre-univex) on X with respect to η, b and ξ.

PROOF. We prove the proposition only for pre-unicave functions, for pre-univex functions the proof is analogous.

Let the hypograph of f, $\mathrm{Hyp}(f)$, be a univex set with respect to η, b and ξ. Then for each $(x, f(x)) \in \mathrm{Hyp}(f)$, $(y, f(y)) \in \mathrm{Hyp}(f)$, and for each $\lambda \in [0,1]$

$$(x + \lambda\eta(x,y), f(x) + \lambda b(x,y,\lambda)\xi(f(y) - f(x))) \in \mathrm{Hyp}(f).$$

It follows that

$$f(x + \lambda\eta(x,y)) \geq f(x) + \lambda b(x,y,\lambda)\xi(f(y) - f(x))$$

for every $\lambda \in [0,1]$. Consequently, by Definition 3.21, (iv), (3.18) and (3.19), function f is a pre-unicave function on X with respect to η, b and ξ. ∎

Now, we give a characterization of some important subclasses of (Φ, Ψ)-concave functions by their upper level sets and hypographs.

PROPOSITION 3.23 *Let X be a nonempty subset of $\mathbf{R}^n$, Φ be a set of mappings φ with $\varphi : X \times X \times [0,1] \to \mathbf{R}^n$ and let Ψ be a set of functions ψ with $\psi : X \times X \times \mathbf{R} \times \mathbf{R} \times [0,1] \to \mathbf{R}$ such that for each $\psi \in \Psi$,*

$$\psi(x,y,\alpha,\beta,\lambda) \geq \min\{\alpha,\beta\} \tag{3.20}$$

for all $x, y \in X, \alpha, \beta \in \mathbf{R}$, $\lambda \in [0,1]$. Moreover, let X be a Φ-convex subset of $\mathbf{R}^n$. If $f : \mathbf{R}^n \to \mathbf{R}$ is (Φ, Ψ)-concave on X, then all upper-level sets $U(f, \delta)$ are Φ-convex subsets of $\mathbf{R}^n$.

PROOF. Let $\delta \in \mathbf{R}$, $x, y \in U(f,\delta)$, $\lambda \in [0,1]$. Then

$$f(x) \geq \delta, f(y) \geq \delta. \tag{3.21}$$

Since X is Φ-convex subset of $\mathbf{R}^n$, there exists $\varphi \in \Phi$ such that $\varphi(x,y,\lambda) \in X$, and, by (Φ, Ψ)-concavity of f on X, we get

$$f(\varphi(x,y,\lambda)) \geq \inf\{\psi(x,y,f(x),f(y),\lambda) \mid \psi \in \Psi\}. \tag{3.22}$$

Combining (3.22), (3.20) and (3.21), we obtain

$$f(\varphi(x,y,\lambda)) \geq \min\{f(x), f(y)\} \geq \delta,$$

hence, $\varphi(x,y,\lambda) \in U(f,\delta)$ proving that $U(f,\delta)$ is Φ-convex. ∎

PROPOSITION 3.24 *Let X be a nonempty Φ-convex subset of $\mathbf{R}^n$, Φ be a set of mappings φ with $\varphi : X \times X \times [0,1] \to \mathbf{R}^n$. Let Ψ consist of a single function ψ, where $\psi : X \times X \times \mathbf{R} \times \mathbf{R} \times [0,1] \to \mathbf{R}$ such that for each $\psi \in \Psi$,*

$$\psi(x, y, \alpha^*, \beta^*, \lambda) \geq \psi(x, y, \alpha, \beta, \lambda) \tag{3.23}$$

for all $x, y \in X$, $\lambda \in [0,1]$, whenever $\alpha^* \geq \alpha$, $\beta^* \geq \beta$. If $f : \mathbf{R}^n \to \mathbf{R}$ is (Φ, Ψ)-concave on X, then the hypograph $\mathrm{Hyp}(f)$ is a $\bar{\Phi}$-convex subset of $\mathbf{R}^{n+1}$, where $\bar{\Phi} = \{\bar{\varphi} \mid \bar{\varphi} = (\varphi, \psi), \varphi \in \Phi\}$.

PROOF. Let $(x, \alpha), (y, \beta) \in \mathrm{Hyp}(f)$ and $\lambda \in [0,1]$. Then, we have $f(x) \geq \alpha$ and $f(y) \geq \beta$. We have to show that there exists a $\bar{\varphi} \in \bar{\Phi}$ such that $\bar{\varphi}((x, \alpha), (y, \beta), \lambda) \in \mathrm{Hyp}(f)$.

Since X is Φ-convex subset of $\mathbf{R}^n$, $x, y \in X$, there exists $\varphi \in \Phi$ such that $\varphi(x, y, \lambda) \in X$, and by (Φ, Ψ)-concavity of f on X, we get

$$f(\varphi(x, y, \lambda)) \geq \psi(x, y, f(x), f(y), \lambda). \tag{3.24}$$

By (3.23) and (3.24) we obtain

$$f(\varphi(x, y, \lambda)) \geq \psi(x, y, \alpha, \beta, \lambda).$$

Setting $\bar{\varphi} = (\varphi, \psi)$, we finally obtain

$$\bar{\varphi}((x, \alpha), (y, \beta), \lambda) \in \mathrm{Hyp}(f).$$

∎

Notice that condition (3.20) is satisfied e.g. for Φ-concave and Φ-quasi-concave functions.

In what follows we shall investigate some properties of local and global extrema of (Φ, Ψ)-concave (convex) functions on Φ-convex sets, we refer to Theorems 3.17 and 3.18. The following two theorems generalize analogical results of [3] and [10].

THEOREM 3.25 *Let X be a nonempty Φ-convex subset of $\mathbf{R}^n$, where Φ is a set of mappings $\varphi : X \times X \times [0,1] \to \mathbf{R}^n$ such that for all $x, y \in X$*

$$\lim_{\lambda \to 0_+} \varphi(x, y, \lambda) = x. \tag{3.25}$$

Let Ψ be a set of functions $\psi : X \times X \times \mathbf{R} \times \mathbf{R} \times [0,1] \to \mathbf{R}$ such that

$$\psi(x, y, \alpha, \beta, \lambda) \geq \min\{\alpha, \beta\} \tag{3.26}$$

for all $x, y \in X$, $\alpha, \beta \in \mathbf{R}$, $\lambda \in [0,1]$. Let $f : \mathbf{R}^n \to \mathbf{R}$ be (Φ, Ψ)-concave on X. If $\bar{x} \in X$ is a strict local maximizer of f over X, then it is a strict global maximizer of f over X.

PROOF. Since $\bar{x} \in X$ is a strict local maximizer, there exists an open ball B with the center at $\bar{x} \in X$ such that $f(x) < f(\bar{x})$ for all $x \in X \cap B$ with $x \neq \bar{x}$.

Suppose that $\bar{x} \in X$ is not a global maximizer. Then there exists $y \in X$ such that

$$f(\bar{x}) < f(y). \tag{3.27}$$

As X is Φ-convex and f is (Φ,Ψ)-concave on X, it follows that there exists $\varphi \in \Phi$ such that $\varphi(\bar{x}, y, \lambda) \in X$ and

$$f(\varphi(\bar{x}, y, \lambda)) \geq \inf\{\psi(\bar{x}, y, f(\bar{x}), f(y), \lambda) \mid \psi \in \Psi\} \tag{3.28}$$

for each $\lambda \in [0, 1]$. By (3.26) we obtain

$$\inf\{\psi(\bar{x}, y, f(\bar{x}), f(y), \lambda) \mid \psi \in \Psi\} \geq \min\{f(\bar{x}), f(y)\} \tag{3.29}$$

and combining (3.28) with (3.29), we get

$$f(\varphi(\bar{x}, y, \lambda)) \geq \min\{f(\bar{x}), f(y)\} \tag{3.30}$$

for each $\lambda \in [0, 1]$. Moreover, from (3.25) we have $\varphi(\bar{x}, y, \lambda_0) \in X \cap B$ for some sufficiently small $\lambda_0 \in (0, 1)$. Let $z = \varphi(\bar{x}, y, \lambda_0)$. Hence, applying (3.27) and (3.30), we conclude that $f(z) \geq f(\bar{x})$, however, we have also $f(z) < f(\bar{x})$, a contradiction. Consequently, $\bar{x}$ must be a strict global maximizer of f over X. ■

Notice that condition (3.25) is satisfied e.g. for (3.18), i.e., for invex sets. Moreover, condition (3.26) is satisfied e.g. for (3.16) and (3.17), hence for Φ-concave, Φ-quasiconcave functions on Φ-convex sets satisfying (3.18).

Clearly, if we drop the assumption of strictness of the local maximizer in Theorem 3.25, then the statement is no longer valid. The semistrict concavity will, however, secure the result.

THEOREM 3.26 *Let X be a nonempty Φ-convex subset of $\mathbf{R}^n$, where Φ is a set of mappings φ with $\varphi : X \times X \times [0,1] \to \mathbf{R}^n$ such that for all $x, y \in X$*

$$\lim_{\lambda \to 0_+} \varphi(x, y, \lambda) = x. \tag{3.31}$$

Let Ψ be a set of functions $\psi : X \times X \times \mathbf{R} \times \mathbf{R} \times [0,1] \to \mathbf{R}$ such that

$$\psi(x, y, \alpha, \beta, \lambda) \geq \min\{\alpha, \beta\}$$

for all $x, y \in X, \alpha, \beta \in \mathbf{R}$, $\lambda \in [0,1]$. Let $f : \mathbf{R}^n \to \mathbf{R}$ be (Φ, Ψ)-semistrictly concave on X. If $\bar{x} \in X$ is a local maximizer of f over X, then it is a global maximizer of f over X.

PROOF. Since $\bar{x} \in X$ is a local maximizer, there exists an open ball B with the center at $\bar{x} \in X$ such that $f(x) \leq f(\bar{x})$ for all $x \in X \cap B$ with $x \neq \bar{x}$.

Suppose that $\bar{x} \in X$ is not a global maximizer. Then there exists $y \in X$ such that

$$f(\bar{x}) < f(y). \tag{3.32}$$

As X is Φ-convex and f is (Φ, Ψ)-semistrictly concave on X, it follows that there exists $\varphi \in \Phi$ such that $\varphi(\bar{x}, y, \lambda) \in X$ and by (3.32)

$$f(\varphi(\bar{x}, y, \lambda)) > \inf\{\psi(\bar{x}, y, f(\bar{x}), f(y), \lambda) \mid \psi \in \Psi\} \tag{3.33}$$

for each $\lambda \in [0, 1]$. By (3.26) we obtain

$$\inf\{\psi(\bar{x}, y, f(\bar{x}), f(y), \lambda) \mid \psi \in \Psi\} \geq \min\{f(\bar{x}), f(y)\} \tag{3.34}$$

and combining (3.33) with (3.34), we get

$$f(\varphi(\bar{x}, y, \lambda)) > \min\{f(\bar{x}), f(y)\} \tag{3.35}$$

for each $\lambda \in (0, 1)$. Moreover, from (3.31) we obtain $\varphi(\bar{x}, y, \lambda_0) \in X \cap B$ for some sufficiently small $\lambda_0 \in (0, 1)$. Let $z = \varphi(\bar{x}, y, \lambda_0)$. Hence, applying (3.32) and (3.35) we conclude that $f(z) > f(\bar{x})$, however, we have also $f(z) \leq f(\bar{x})$, a contradiction. ■

Clearly, analogous theorems to Theorems 3.25 and 3.26 can be formulated and proved for local and global minimizers and (Φ, Ψ)-convex functions.

EXAMPLE 3.27 Let Φ be a set of functions $\varphi^{(k,m)}$ with $\varphi^{(k,m)} : \mathbf{R}^2 \times \mathbf{R}^2 \times [0, 1] \to \mathbf{R}^2$, such that $\varphi^{(k,m)} = \left(\varphi_1^{(k)}, \varphi_2^{(m)}\right)$, $k, m \in \{1, 2, \ldots, K\}$ and

$$\varphi_i^{(k)}(x, y, \lambda) = x_i + \lambda^k(y_i - x_i), \quad i = 1, 2,$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$. Let X be a nonempty Φ-convex subset of $\mathbf{R}^2$ and let Ψ consist of K functions, $\Psi = \{\psi_k \mid k = 1, 2, \ldots, K\}$, where

$$\psi_k(x, y, \alpha, \beta, \lambda) = \alpha + \lambda^k(\beta - \alpha)$$

for all $x, y \in X$, $\alpha, \beta \in \mathbf{R}$, $\lambda \in [0, 1]$. Then by Definition 2.73 we obtain particular classes of Φ-convex sets and by Definition 3.21, (Φ, Ψ)-concave and (Φ, Ψ)-convex functions. Observe that X is Φ-convex if any two points $x, y \in X$ can be connected by a curve such that $\varphi^{(k,m)}(x, y, \lambda)$ belongs to X for each $\lambda \in [0, 1]$ and some $k, m \in \{1, 2, \ldots, K\}$. For illustration, in Figure 3.4, we have depicted $\varphi^{(1,4)} = (\varphi_1^{(1)}, \varphi_2^{(4)})$, where

$$\begin{aligned} \varphi_1^{(1)}(x, y, \lambda) &= x_1 + \lambda(y_1 - x_1), \\ \varphi_2^{(4)}(x, y, \lambda) &= x_2 + \lambda^4(y_2 - x_2), \end{aligned}$$

and $\varphi^{(3,2)} = (\varphi_1^{(3)}, \varphi_2^{(2)})$, where

$$\begin{aligned}\varphi_1^{(3)}(x,y,\lambda) &= x_1 + \lambda^3(y_1 - x_1),\\ \varphi_2^{(2)}(x,y,\lambda) &= x_2 + \lambda^2(y_2 - x_2)\end{aligned}$$

for all $\lambda \in [0,1]$, where $x = (1,-2), y = (-2,2)$. □

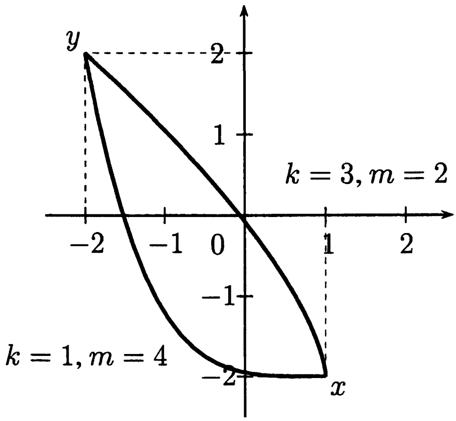

Figure 3.4.

4. Differentiable Functions

Let us turn now to differentiable generalized concave functions. We start with a brief review of the results known for differentiable concave and quasiconcave functions; see also [3].

4.1. Differentiable Quasiconcave Functions

First let us recall that a real-valued function f defined on an open subset X of $\mathbf{R}^n$ is called *continuously differentiable* on X, if f has all partial derivatives of the first order and all of them are continuous on X. The vector $\nabla f(x_0)$ defined for $x_0 \in X$ by

$$\nabla f(x_0) = \left(\frac{\partial f(x_0)}{\partial x_1}, \frac{\partial f(x_0)}{\partial x_2}, \ldots, \frac{\partial f(x_0)}{\partial x_n}\right)$$

is then called the *gradient of f at x_0*.

If f has all partial derivatives of the second order and all of them are continuous on X, then we say that f is *twice continuously differentiable on X*. The

matrix

$$\nabla^2 f(x_0) = \begin{bmatrix} \dfrac{\partial^2 f(x_0)}{\partial x_1 \partial x_1} & \cdots & \dfrac{\partial^2 f(x_0)}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial^2 f(x_0)}{\partial x_n \partial x_1} & \cdots & \dfrac{\partial^2 f(x_0)}{\partial x_n \partial x_n} \end{bmatrix}$$

is then symmetric $n \times n$ matrix called the *Hessian matrix of f at x_0*. Occasionally we need to multiply the Hessian of f at x by a vector from $\mathbf{R}^n$, say y. If so, we use the notation $\nabla^2 f(x)y$.

Now, we begin with a brief review of known results. The following result gives a characterization of differentiable concave functions; see [3].

PROPOSITION 3.28 *Let X be an open convex subset of $\mathbf{R}^n$ and $f : X \to \mathbf{R}$ be a function differentiable on X. Then f is*

(i) *concave if and only if*

$$f(y) - f(x) \leq \langle \nabla f(x), y - x \rangle \tag{3.36}$$

holds for every $x, y \in X$;

(ii) *strictly concave if and only if*

$$f(y) - f(x) < \langle \nabla f(x), y - x \rangle \tag{3.37}$$

holds for every $x, y \in X$, $x \neq y$;

(iii) *convex if and only if*

$$f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle \tag{3.38}$$

holds for every $x, y \in X$;

(iv) *strictly convex if and only if*

$$f(y) - f(x) > \langle \nabla f(x), y - x \rangle \tag{3.39}$$

holds for every $x, y \in X$, $x \neq y$.

The following proposition is a well known characterization of concave functions by the second derivatives, i.e., by the Hessian matrix; see e.g. [3].

PROPOSITION 3.29 *Let X be an open convex subset of $\mathbf{R}^n$ and $f : X \to \mathbf{R}$ be a twice-continuously differentiable function on X. Then f is concave on X if and only if its Hessian matrix is negative semidefinite for every $x \in X$, that is, for every $x \in X$ and every $y \in \mathbf{R}^n$ it holds*

$$\langle \nabla^2 f(x)y, y \rangle \leq 0.$$

If $\nabla^2 f(x)$ is negative definite for every $x \in X$, then f is strictly concave on X.

The last statement in Proposition 3.29 cannot be reversed, since there exist strictly concave functions whose Hessians are not negative definite, e.g., $f(x) = 1 - \|x\|^2$ at $x = 0$.

The differentiable quasiconcave functions can be characterized by the following proposition; see [3].

PROPOSITION 3.30 *Let X be an open convex subset of $\mathbf{R}^n$ and $f : X \to \mathbf{R}$ be a function differentiable on X. Then*

(i) *f is quasiconcave if and only if*

$$f(y) - f(x) \geq 0 \textit{ implies that } \langle \nabla f(x), y - x \rangle \geq 0 \tag{3.40}$$

holds for every $x, y \in X$;

(ii) *f is quasiconvex if and only if*

$$f(y) - f(x) \leq 0 \textit{ implies that } \langle \nabla f(x), y - x \rangle \leq 0 \tag{3.41}$$

holds for every $x, y \in X$.

Notice that (3.36) implies (3.40), and (3.38) implies (3.41). For quasimonotonicity we have the following characterization; see [71].

PROPOSITION 3.31 *Let X be an open convex subset of $\mathbf{R}^n$ and f be a differentiable function on X, $f : X \to \mathbf{R}$. Then f is quasimonotone if and only if*

$$f(y) - f(x) \geq 0 \textit{ implies that } \langle \nabla f(z), y - x \rangle \geq 0$$

holds for every $x, y \in X$ and every $z = \lambda x + (1 - \lambda)y$, where $\lambda \in (0, 1)$.

The detailed treatment of (strictly, semistrictly) differentiable quasiconcave functions can be found in [3].

4.2. Pseudoconcave Functions

Replacing inequality relations in (3.40) (or (3.41)) by the strict inequalities, we do not obtain the strict quasiconcavity (strict quasiconvexity) of f, as it could be guessed. In fact, we obtain a new class of functions located between concave and quasiconcave functions (convex and quasiconvex functions). These functions are called pseudoconcave (pseudoconvex) functions; see [3].

DEFINITION 3.32 *Let X be an open convex subset of $\mathbf{R}^n$, and $f : X \to \mathbf{R}$ be a function differentiable on X. Then f is called*

(i) pseudoconcave on X (PCA) *if*

$$f(y) - f(x) > 0 \textit{ implies that } \langle \nabla f(x), y - x \rangle > 0 \tag{3.42}$$

for every $x, y \in X$;

(ii) strictly pseudoconcave on X *if*

$$f(y) - f(x) \geq 0 \textit{ implies that } \langle \nabla f(x), y - x \rangle > 0 \tag{3.43}$$

for every $x, y \in X, x \neq y$;

(iii) pseudoconvex on X (PCV) *if*

$$f(y) - f(x) < 0 \textit{ implies that } \langle \nabla f(x), y - x \rangle < 0 \tag{3.44}$$

for every $x, y \in X$;

(iv) strictly pseudoconvex on X *if*

$$f(y) - f(x) \leq 0 \textit{ implies that } \langle \nabla f(x), y - x \rangle < 0 \tag{3.45}$$

for every $x, y \in X, x \neq y$.

From Definition 3.32 we immediately obtain an important property of pseudoconcave (pseudoconvex) functions.

THEOREM 3.33 *Let X be an open convex subset of $\mathbf{R}^n$, $f : X \to \mathbf{R}$ be a differentiable (strictly) pseudoconcave function on X. If $\nabla f(x^*) = 0$ for some $x^* \in X$, then x^* is a (strict) global maximizer of f over X. Likewise, if f is a differentiable (strictly) pseudoconvex function on X such that $\nabla f(x^*) = 0$ holds for some $x^* \in X$, then x^* is a (strict) global minimizer of f over X.*

PROOF. We prove only the first part of the theorem for (strictly) pseudoconcave functions, the rest of the proof is obvious.

By (3.42), resp. (3.44), $\nabla f(x^*) = 0$ implies $f(x^*) \geq f(x)$, resp. $f(x^*) > f(x)$, for all $x \in X$, and $x^*$is a (strict) global maximizer of f over X. ■

The following two propositions give a characterization of continuously and twice continuously differentiable (strictly) pseudoconcave functions; see [3].

PROPOSITION 3.34 *Let X be an open convex subset of $\mathbf{R}^n$, $f : X \to \mathbf{R}$ be a continuously differentiable function on X. Then f is (strictly) pseudoconcave on X if and only if for every $x_0 \in X$ and $y \in \mathbf{R}^n$ such that $\|y\| = 1$ and $\langle \nabla f(x_0), y \rangle = 0$ the function $F(t) = f(x_0 + ty)$ attains a (strict) local maximum at $t = 0$.*

The geometric interpretation of the condition in Proposition 3.34 is that the function f, restricted to the line passing through x_0 in the direction y orthogonal to the gradient of f at x_0, has a local maximum at x_0.

PROPOSITION 3.35 *Let X be an open convex subset of $\mathbf{R}^n$, $f : X \to \mathbf{R}$ be a twice-continuously differentiable function on X. Then f is (strictly) pseudoconcave on X if and only if for every $x_0 \in X$ and $y \in \mathbf{R}^n$ such that $\|y\| = 1$ and $\langle \nabla f(x_0), y\rangle = 0$ either $\langle \nabla^2 f(x_0)y, y\rangle < 0$ or $\langle \nabla^2 f(x_0)y, y\rangle = 0$ and the function $F(t) = f(x_0 + ty)$ attains a (strict) local maximum at $t = 0$.*

The analogical propositions to Propositions 3.34 and 3.35 hold for (strictly) pseudoconvex functions. We left their formulation to the reader.

Pseudoconcave functions are intermediate between concave and semistrictly quasiconcave functions. This result will be proved in the following theorem together with some other inclusions between the classes of generalized concave (convex) functions.

THEOREM 3.36 *Let X be an open convex subset of $\mathbf{R}^n$, $f : X \to \mathbf{R}$ be a continuously differentiable function on X. Then the following chains of implications hold:*

(i) *If f is CA on X, then f is PCA on X;*

(ii) *if f is PCA on X, then f is semistrictly QCA on X;*

(iii) *if f is semistrictly QCA on X, then f is QCA on X;*

(iv) *if f is QCA on X, then f is US on X;*

(v) *if f is US on X, then f is UQCN on X.*

PROOF.

(i) Let f be concave on X. Then by (3.36) we have

$$f(y) - f(x) \le \langle \nabla f(x), y - x\rangle$$

for every $x, y \in X$. If $0 < f(y) - f(x)$, then by (3.42)

$$0 < \langle \nabla f(x), y - x\rangle,$$

hence, by (3.42), f is pseudoconcave on X.

(ii) Let f be pseudoconcave on X, let $x, y \in X$, $x \neq y$.

First, we show that f is quasiconcave on X. By Proposition 3.30 this is equivalent to the following implication

$$f(y) - f(x) \ge 0 \text{ implies that } \langle \nabla f(x), y - x\rangle \ge 0. \qquad (3.46)$$

If $f(y) > f(x)$, then by (3.42), implication (3.46) holds.

Suppose, on contrary, that

$$f(x) = f(y) \text{ and } \langle \nabla f(x), y - x \rangle < 0. \tag{3.47}$$

Setting

$$F(t) = f(x + t(y - x)), \quad t \in [0, 1],$$

there exists $t_0 \in (0, 1)$ such that t_0 is a local minimizer of F. Therefore at $z = x + t_0(y - x)$ we obtain

$$\langle \nabla f(z), y - z \rangle = 0. \tag{3.48}$$

However, by (3.47) we have $F(t_0) = f(z) < f(y)$. Since f is pseudoconcave, it follows by (3.42) $\langle \nabla f(z), y - z \rangle > 0$, a contradiction to (3.48). Consequently, f is quasiconcave on X.

To prove that f is semistrictly quasiconcave on X, consider $f(y) > f(x)$ and $u = x + \lambda(y - x)$, $\lambda \in (0, 1)$. Then by (3.42) we obtain

$$\langle \nabla f(x), y - x \rangle > 0.$$

Using again the proof by contradiction, we easily conclude that

$$f(u) > f(x) = \min\{f(x), f(y)\}.$$

Hence, semistrict quasiconcavity follows.

(iii) This inclusion follows directly from Definition 3.2.

(iv) All upper-level sets $U(f, \gamma)$ are convex and, therefore, starshaped.

(v) This inclusion is also clear from Definition 3.15. Indeed, for the path $\mathbf{P}(x, y)$ we can take the union $\mathbf{I}(x, z) \cup \mathbf{I}(z, y)$ of the two line segments, where $z \in \text{Ker}(U(f, \gamma))$ and $\gamma = \min\{f(x), f(y)\}$.

∎

An analogous theorem is valid for generalized convex functions.

4.3. Incave, Pseudoincave and Pseudounicave Functions

We start this subsection with a generalization of the concept of differentiable concave (convex) functions based on Proposition 3.28. After that we shall generalize the pseudoconcave (pseudoconvex) function from Definitions 3.32. In fact, we make some rearrangements of the formulae (3.36) - (3.39) and (3.42) - (3.45) to define new and broader classes of differentiable functions with some concavity properties. Here, the new classes are called incave, pseudoincave,

unicave and pseudounicave functions. Our approach is motivated by the paper [10], our terminology is, however, different from that used in [10]. Here, we reflect the new classes of functions, namely (Φ, Ψ)-concave functions, introduced in Definition 3.21.

DEFINITION 3.37 *Let X be an open convex subset of $\mathbf{R}^n$, $f : X \to \mathbf{R}$ be a differentiable function on X. Let η be a mapping of $X \times X$ into $\mathbf{R}^n$. Then f is called*

(i) incave on X with respect to η (IA) *if*

$$f(y) - f(x) \leq \langle \nabla f(x), \eta(x, y) \rangle$$

for every $x, y \in X$;

(ii) strictly incave on X with respect to η *if*

$$f(y) - f(x) < \langle \nabla f(x), \eta(x, y) \rangle$$

for every $x, y \in X, x \neq y$;

(iii) invex on X with respect to η (IV) *if*

$$f(y) - f(x) \geq \langle \nabla f(x), \eta(x, y) \rangle$$

for every $x, y \in X$;

(iv) strictly invex on X with respect to η *if*

$$f(y) - f(x) > \langle \nabla f(x), \eta(x, y) \rangle$$

for every $x, y \in X, x \neq y$.

In a special case, namely, if $\eta(x, y) = y - x$, we obtain by Proposition 3.28 that any incave (invex) function with respect to η is concave (convex). The class of incave (invex) functions can be further extended by the following definition.

DEFINITION 3.38 *Let X be an open convex subset of $\mathbf{R}^n$, $f : X \to \mathbf{R}$ be a differentiable function on X. Let η be a mapping of $X \times X$ into $\mathbf{R}^n$. Then f is called*

(i) pseudoincave on X with respect to η (PIA) *if*

$$f(y) - f(x) > 0 \textit{ implies that } \langle \nabla f(x), \eta(x, y) \rangle > 0$$

for every $x, y \in X$;

(ii) strictly pseudoincave on X with respect to η *if*

$$f(y) - f(x) \geq 0 \textit{ implies that } \langle \nabla f(x), \eta(x,y) \rangle > 0$$

for every $x, y \in X, x \neq y$;

(iii) pseudoinvex on X with respect to η (PIV) *if*

$$f(y) - f(x) < 0 \textit{ implies that } \langle \nabla f(x), \eta(x,y) \rangle < 0$$

for every $x, y \in X$;

(iv) strictly pseudoinvex on X with respect to η *if*

$$f(y) - f(x) \leq 0 \textit{ implies that } \langle \nabla f(x), \eta(x,y) \rangle < 0$$

for every $x, y \in X, x \neq y$.

Notice that the differentiable incave functions are pseudoincave. Further, we define unicave (univex) and pseudounicave (pseudounivex) functions.

DEFINITION 3.39 *Let* X *be an open convex subset of* $\mathbf{R}^n$, $f : X \to \mathbf{R}$ *be a differentiable function on* X. *Let* η, b *and* ξ *be functions,* $\eta : X \times X \to \mathbf{R}^n$, $b : X \times X \to [0, +\infty)$, $\xi : \mathbf{R} \to \mathbf{R}$. *Then* f *is called*

(i) unicave on X with respect to η, b and ξ (UA) *if*

$$b(x,y)\xi(f(y) - f(x)) \leq \langle \nabla f(x), \eta(x,y) \rangle$$

for every $x, y \in X$;

(ii) strictly unicave on X with respect to η, b and ξ *if*

$$b(x,y)\xi(f(y) - f(x)) < \langle \nabla f(x), \eta(x,y) \rangle$$

for every $x, y \in X, x \neq y$;

(iii) univex on X with respect to η, b and ξ (UV) *if*

$$b(x,y)\xi(f(y) - f(x)) \geq \langle \nabla f(x), \eta(x,y) \rangle$$

for every $x, y \in X$;

(iv) strictly univex on X with respect to η, b and ξ *if*

$$b(x,y)\xi(f(y) - f(x)) > \langle \nabla f(x), \eta(x,y) \rangle$$

for every $x, y \in X, x \neq y$.

Clearly, if $\eta(x,y) = y - x$, $b(x,y) = 1$, $\xi(t) = t$ for all $x, y \in X$, $t \in \mathbf{R}$, then by Definition 3.37 a unicave (univex) function on X with respect to η, b and ξ is incave (invex) on X with respect to η.

DEFINITION 3.40 *Let X be an open convex subset of $\mathbf{R}^n$, $f : X \to \mathbf{R}$ be a differentiable function on X. Let η, b and ξ be functions, $\eta : X \times X \to \mathbf{R}^n$, $b : X \times X \to [0, +\infty)$, $\xi : \mathbf{R} \to \mathbf{R}$. Then f is called*

(i) pseudounicave on X with respect to η, b and ξ (PUA) *if*

$$b(x,y)\xi\left(f(y) - f(x)\right) > 0 \textit{ implies that } \langle \nabla f(x), \eta(x,y) \rangle > 0$$

for every $x, y \in X$;

(ii) strictly pseudounicave on X with respect to η, b and ξ *if*

$$b(x,y)\xi\left(f(y) - f(x)\right) \geq 0 \textit{ implies that } \langle \nabla f(x), \eta(x,y) \rangle > 0$$

for every $x, y \in X$, $x \neq y$;

(iii) pseudounivex on X with respect to η, b and ξ (PUV) *if*

$$b(x,y)\xi\left(f(y) - f(x)\right) < 0 \textit{ implies that } \langle \nabla f(x), \eta(x,y) \rangle < 0$$

for every $x, y \in X$;

(iv) strictly pseudounivex on X with respect to η, b and ξ *if*

$$b(x,y)\xi\left(f(y) - f(x)\right) \leq 0 \textit{ implies that } \langle \nabla f(x), \eta(x,y) \rangle < 0$$

for every $x, y \in X$, $x \neq y$.

Evidently, any unicave (univex) function on X with respect to η, b and ξ is pseudounicave (pseudounivex) on X with respect to η, b and ξ.

In particular, if $\eta(x,y) = y - x$, $b(x,y) = 1$, $\xi(t) = t$ for all $x, y \in X$, $t \in \mathbf{R}$, then by Definition 3.39, any unicave (univex) function on X with respect to η, b and ξ is pseudoconcave (pseudoconvex) on X. By (3.40) it is also quasiconcave (quasiconvex) on X.

Moreover, if f is incave on X with respect to some η, then by (i) of Definition 3.37, f is unicave on X with respect to η, b and ξ, where $b(x,y) = 1$ for all $x, y \in X$ and $\xi(t) = t$, for all $t \in \mathbf{R}$, thus f is also pseudounicave on X with respect to η, b and ξ.

If η and η^* are different functions then the class of all incave functions on X with respect to η is generally different from the class of all incave functions on X with respect to η^*. Therefore, the classes of incave functions corresponding to different η's cannot be linearly ordered by inclusion. The analogical conclusion holds also for pseudoincave functions. Moreover, by analogy, the same conclusion holds for unicave and pseudounicave functions, too.

In (iv), section 3.2, see (3.18), (3.19), we have defined the pre-unicave functions without any differentiability property. The next proposition gives some sufficient conditions for the differentiable functions to be unicave.

PROPOSITION 3.41 *Let X be an open convex subset of $\mathbf{R}^n$, f be differentiable and (strictly) pre-unicave on X with respect to η, b and ξ. Let* $\lim_{\lambda\to 0_+} b(x,y,\lambda)$ *exist for all $x, y \in X$ and let $\bar{b}(x,y)$ be defined by*

$$\bar{b}(x,y) = \lim_{\lambda\to 0_+} b(x,y,\lambda)$$

for all $x, y \in X$. Then f is (strictly) unicave on X with respect to η, $\bar{b}$ and ξ.

PROOF. We prove the assertion only for the pre-unicave functions, for strictly pre-unicave ones the proof is analogical.

Let f be differentiable and pre-unicave on X with respect to η, b and ξ. Then by (3.18), (3.19), we obtain for all $x, y \in X$ and $\lambda \in [0,1]$

$$f(x+\lambda\eta(x,y)) \geq f(x) + \lambda b(x,y,\lambda)\xi\left(f(y)-f(x)\right),$$

which can be rearranged as

$$\frac{f(x+\lambda\eta(x,y))-f(x)}{\lambda} \geq b(x,y,\lambda)\xi\left(f(y)-f(x)\right), \tag{3.49}$$

Letting $\lambda \to 0_+$, we obtain from (3.49)

$$\langle\nabla f(x), \eta(x,y)\rangle \geq \bar{b}(x,y)\xi\left(f(y)-f(x)\right).$$

The last inequality is exactly (i) of Definition 3.39, hence f is unicave on X with respect to η, $\bar{b}$ and ξ. ■

By analogy, a similar proposition to Proposition 3.35 can be formulated and proved for pre-univex function. A verification of this statement is left to the reader.

5. Constrained Optimization

Results of Section 3.3 guarantee that local maximizers of some generalized concave functions are also global maximizers. In this section we present further results that can be useful in the analysis and solution procedures of optimization problems. Applications to fuzzy optimization problems are discussed in detail in Part II.

Let f and $g_1, g_2, \ldots, g_m$ be real-valued functions defined on $\mathbf{R}^n$, and let X be the subset of $\mathbf{R}^n$ defined by

$$X = \{x \in \mathbf{R}^n \mid g_i(x) \geq 0,\ i = 1, 2, \ldots, m\}. \tag{3.50}$$

We consider the following optimization problem:

$$\begin{array}{ll} \text{maximize} & f(x) \\ \text{subject to} & x \in X, \end{array} \tag{3.51}$$

In this context, the function f and functions $g_1, g_2, \dots, g_m$ are called the objective function and the constraint functions, respectively. The elements of X are called feasible solutions, the global maximizers of f over X are called optimal solutions, and the (strict) local maximizers are called (strict) local optimal solutions. The same terminology is used for minimization problems.

PROPOSITION 3.42 *Let Φ consist of a single mapping φ with $\varphi : \mathbf{R}^n \times \mathbf{R}^n \times [0,1] \to \mathbf{R}^n$, Ψ be a given set of functions $\psi : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \times [0,1] \to \mathbf{R}$ such that for each $\psi \in \Psi$,*

$$\psi(x, y, \alpha, \beta, \lambda) \geq \min\{\alpha, \beta\}$$

for all $x, y \in \mathbf{R}^n$, $\alpha, \beta \in \mathbf{R}$, $\lambda \in [0,1]$. Let $g_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, 2, \dots, m$, be (Φ, Ψ)-concave functions on $\mathbf{R}^n$. Then the set of feasible solutions X defined by (3.50) is Φ-convex.

PROOF. By Proposition 3.23 applied to g_i and $\delta = 0$, all upper-level sets $U(g_i, 0)$ are Φ-convex, $i = 1, 2, \dots, m$. By the definition of upper-level set and (3.50), we obtain

$$X = \bigcap_{i=1}^{m} U(g_i, 0). \tag{3.52}$$

Since $\Phi = \{\varphi\}$, it is clear that the intersection (3.52) of m Φ-convex sets is Φ-convex, too. ∎

Notice that we do not require any assumption about φ in Proposition 3.42. If η is a mapping of $X \times X$ into X and $\varphi : \mathbf{R}^n \times \mathbf{R}^n \times [0,1] \to \mathbf{R}^n$ is defined by $\varphi(x, y, \lambda) = x + \lambda\eta(x, y)$, then Proposition 3.42 guarantees that X is invex.

THEOREM 3.43 *Let Φ consist of a single mapping $\varphi : \mathbf{R}^n \times \mathbf{R}^n \times [0,1] \to \mathbf{R}^n$ such that for all $x, y \in \mathbf{R}^n$*

$$\lim_{\lambda \to 0+} \varphi(x, y, \lambda) = x.$$

Let Ψ be a set of functions $\psi : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \times [0,1] \to \mathbf{R}$ such that for each $\psi \in \Psi$,

$$\psi(x, y, \alpha, \beta, \lambda) \geq \min\{\alpha, \beta\}$$

for all $x, y \in \mathbf{R}^n$, $\alpha, \beta \in \mathbf{R}$, $\lambda \in [0,1]$. Let all functions $f : \mathbf{R}^n \to \mathbf{R}$, $g_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, 2, \dots, m$, be (Φ, Ψ)-concave on $\mathbf{R}^n$. If x^* is a unique local optimal solution of (3.51), then x^* is a unique optimal solution of (3.51).

PROOF. By Proposition 3.42, the set of feasible solutions X is Φ-convex. Now, applying the terminology introduced at the beginning of this section, the proof follows directly from Theorem 3.25. ■

In the following theorem we remove the uniqueness of the (local) optimal solution and replace it by a stronger concavity assumption about the objective function.

THEOREM 3.44 *Let Φ consist of a single mapping $\varphi : \mathbf{R}^n \times \mathbf{R}^n \times [0,1] \to \mathbf{R}^n$ such that for all $x, y \in \mathbf{R}^n$*

$$\lim_{\lambda \to 0_+} \varphi(x,y,\lambda) = x.$$

Let Ψ be a set of functions $\psi : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \times [0,1] \to \mathbf{R}$ such that for each $\psi \in \Psi$,

$$\psi(x,y,\alpha,\beta,\lambda) \geq \min\{\alpha,\beta\}$$

for all $x, y \in \mathbf{R}^n$, $\alpha, \beta \in \mathbf{R}$, $\lambda \in [0,1]$. Let $f : \mathbf{R}^n \to \mathbf{R}$ be semistrictly (Φ,Ψ)-concave on $\mathbf{R}^n$ and the functions $g_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1,2,\ldots,m$, be (Φ,Ψ)-concave on $\mathbf{R}^n$. If x^* is a local optimal solution of (3.51), then x^* is an optimal solution of (3.51).

PROOF. By Proposition 3.42, the set of feasible solutions X is Φ-convex. The rest of the theorem follows directly from Theorem 3.26. ■

The same assertion can be obtained under some different assumptions; see [82].

THEOREM 3.45 *Let η, b and ξ be functions, $\eta : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, $b : \mathbf{R}^n \times \mathbf{R}^n \times [0,1] \to (0,+\infty)$, $\xi : \mathbf{R} \to \mathbf{R}$, where ξ is strictly increasing with $\xi(0) = 0$. Let both Φ and Ψ contain a single function, i.e. $\Phi = \{\varphi\}$ and $\Psi = \{\psi\}$, where*

$$\varphi(x,y,\lambda) = x + \lambda\eta(x,y)$$

and

$$\psi(x,y,\alpha,\beta,\lambda) = \alpha + \lambda b(x,y,\lambda)\xi(\beta-\alpha)$$

for all $x, y \in X$, $\alpha, \beta \in \mathbf{R}$, $\lambda \in [0,1]$. Let X be Φ-convex. Let $f : \mathbf{R}^n \to \mathbf{R}$ and $g_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1,2,\ldots,m$, be (Φ,Ψ)-concave on $\mathbf{R}^n$. If x^* is a local optimal solution of (3.51), then x^* is an optimal solution of (3.51).

PROOF. Since $x^* \in X$ is a local maximizer, there exists an open ball B with the center at $x^* \in X$, such that

$$f(x) \leq f(x^*) \tag{3.53}$$

for all $x \in X \cap B$, $x \neq x^*$.

Suppose that $x^* \in X$ is not a global maximizer. Then there exists $y \in X$, $x^* \neq y$, such that

$$f(x^*) < f(y). \tag{3.54}$$

As X is Φ-convex and f is (Φ, Ψ)-concave on X, it follows that

$$x^* + \lambda\eta(x^*, y) \in X$$

and, by (3.32), for each $\lambda \in (0, 1]$

$$f(x^* + \lambda\eta(x^*, y)) \geq f(x^*) + \lambda b(x, y, \lambda)\xi(f(y) - f(x^*)). \tag{3.55}$$

Since ξ is strictly increasing with $\xi(0) = 0$, $\lambda b(x^*, y, \lambda) > 0$, we obtain, from (3.54) and (3.55),

$$\lambda b(x^*, y, \lambda)\xi(f(y) - f(x^*)) > 0, \tag{3.56}$$

and, by combining (3.55) with (3.56), we get

$$f(x^* + \lambda\eta(x^*, y)) > f(x^*) \tag{3.57}$$

for each $\lambda \in (0, 1)$. Moreover, from (3.57) we obtain $x^* + \lambda\eta(x^*, y_0) \in X \cap B$ for some sufficiently small $\lambda_0 \in (0, 1)$. Let $z = x^* + \lambda\eta(x^*, y_0)$. Then applying (3.53) and (3.57) we conclude that $f(z) > f(x^*)$, however, we have also $f(z) \leq f(x^*)$, a contradiction. ■

In paragraph (iv), Section 3.3, the (Φ, Ψ)-concave functions f, g_i from Theorem 3.45 have been named pre-uniconcave functions.

Notice that in the preceding theorems we have not required differentiability of the functions f and g_i. The following theorem can be derived for differentiable generalized concave functions as follows; see also [82].

THEOREM 3.46 *Let all f, g_i be differentiable functions on $\mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}$, $g_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, 2, \ldots, m$. Let η, b_i and ξ_i be functions such that $\eta : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, $b_i : \mathbf{R}^n \times \mathbf{R}^n \to [0, +\infty)$, $\xi_i : \mathbf{R} \to \mathbf{R}$, $i = 0, 1, 2, \ldots, m$. Let f be unicave on some open convex set containing the set of feasible solutions X with respect to η, b_0 and let ξ_0, let ξ_0 be strictly increasing, $\xi_0(0) = 0$. Let g_i be unicave on some open convex set containing the set of feasible solutions X with respect to η, b_i and ξ_i, where ξ_i are superlinear functions, i.e., for every $\alpha, \beta \in \mathbf{R}$, $x, y \in \mathbf{R}^n$, it holds*

$$\xi_i(\alpha x + \beta y) \geq \alpha\xi_i(x) + \beta\xi_i(y), \quad i = 1, 2, \ldots, m.$$

If there exist $x^*, y^* \in \mathbf{R}^n$ satisfying the following conditions

$$\nabla f(x^*) + (\nabla g_1(x^*), \ldots, \nabla g_m(x^*))\, y^* = \theta, \tag{3.58}$$

$$y_i^* g_i(x^*) = 0, \quad i = 1, 2, \ldots, m, \tag{3.59}$$
$$g_i(x^*) \geq 0, \quad i = 1, 2, \ldots, m, \tag{3.60}$$
$$y^* \geq \theta, \tag{3.61}$$
$$b_0(x, x^*) > 0 \quad \text{for each } x \in X, \tag{3.62}$$
$$b_i(x, x^*) \geq 0 \quad \text{for each } x \in X, i = 1, 2, \ldots, m, \tag{3.63}$$

then x^* is an optimal solution of (3.51).

PROOF. Suppose that x^* satisfies (3.58) - (3.63) but is not an optimal solution of (3.51), that is, we suppose that $x^* \in X$ is not a global maximizer. Then there exists $x' \in X$, $x^* \neq x'$, such that

$$f(x') - f(x^*) > 0. \tag{3.64}$$

Since ξ_0 is strictly increasing, $\xi_0(0) = 0$, we obtain from (3.64)

$$\xi_0(f(x') - f(x^*)) > 0.$$

This, along with (3.62), yields

$$b_0(x', x^*)\xi_0(f(x') - f(x^*)) > 0. \tag{3.65}$$

Since f is unicave on some open convex set containing the set of feasible solution X with respect to η, b_0 and ξ_0, by (i) of Definition 3.39, we have

$$b_0(x', x^*)\xi_0(f(x') - f(x^*)) \leq \langle \nabla f(x^*), \eta(x^*, x') \rangle.$$

Substituting here from (3.58), and using the fact that, for each $i = 1, 2, \ldots, m$, g_i are unicave on some open convex set containing the set of feasible solutions X with respect to η, b_i and ξ_i, we get

$$\begin{aligned} b_0(x', x^*)\xi_0(f(x') - f(x^*)) &\leq -\sum_{i=1}^{m} y_i^* \langle \nabla g_i(x^*), \eta(x^*, x') \rangle \qquad (3.66) \\ &\leq -\sum_{i=1}^{m} y_i^* b_i(x', x^*)\xi_i(g_i(x') - g_i(x^*)). \end{aligned}$$

Since ξ_i are superlinear functions,

$$\xi_i(y_i^* g_i(x') - y_i^* g_i(x^*)) \geq \xi_i(y_i^* g_i(x')) + \xi_i(-y_i^* g_i(x^*)) \tag{3.67}$$

and by (3.59) we obtain

$$\xi_i(-y_i^* g_i(x^*)) = 0. \tag{3.68}$$

Moreover,

$$\xi_i(y_i^* g_i(x') - y_i^* g_i(x^*)) = y_i^* \xi_i(g_i(x') - g_i(x^*)). \tag{3.69}$$

From (3.67) - (3.69), we have

$$-y_i^* b_i(x', x^*)\xi_i(g_i(x') - g_i(x^*)) \leq b_i(x', x^*)\xi_i(-y_i^* g_i(x'))$$

for each $i = 1, 2, \ldots, m$. Since $y_i^* \geq 0$ and $g_i(x') \leq 0$, we know that

$$-y_i^* b_i(x', x^*)\xi_i(g_i(x') - g_i(x^*)) \leq 0. \tag{3.70}$$

Consequently, by (3.66) and (3.70), we obtain

$$b_0(x', x^*)\xi_0(f(x') - f(x^*)) \leq 0,$$

a contradiction to (3.65), hence the result follows. ■

Some other results for differentiable univex functions, namely concerning duality, can be found in [3], [10], [25], [59] or [82].

Chapter 4

TRIANGULAR NORMS AND T-QUASICONCAVE FUNCTIONS

In this chapter we deal with a subclass of upper and lower-starshaped functions, particularly real-valued functions whose ranges are subsets of the unit interval $[0, 1]$. The reason for this restriction comes from applications to real-world problems. There exist many practical situations, e.g., in decision making, economics and business, and also in technical or technological disciplines, where such functions play an essential role. These applications will be dealt with in Part II.

In this chapter we are interested in other type of generalization of quasiconcave and quasiconvex functions than those already defined in the preceding chapter. In what follows, generalizations are based on triangular norms and conorms and the new functions are called T-quasiconcave and S-quasiconvex, respectively. Later on, we shall see how T-quasiconcave and S-quasiconvex functions are related to the generalized concave functions defined in the preceding chapter.

1. Triangular Norms and Conorms

The notion of a triangular norm was introduced by Schweizer and Sklar in their development of a probabilistic generalization of the theory of metric spaces. This development was initiated by K. Menger [72], who proposed to replace the distance $d(x, y)$ between points x and y of a metric space by a real-valued function F_{xy} of a real variable whose value $F_{xy}(\alpha)$ is interpreted as the probability that the distance between x and y is less than α. This interpretation leads to straightforward generalizations of all requirements of the standard definition of a metric except for that of the triangular inequality. Elaborating Menger's idea, Schweizer and Sklar [113] proposed to replace the triangular

inequality by the inequality

$$F_{xy}(\alpha+\beta) \geq T(F_{xy}(\alpha), F_{yz}(\beta))$$

where T is a function from $[0,1]^2$ into $[0,1]$ satisfying the following conditions (4.1) - (4.4).

DEFINITION 4.1 *Let $T : [0,1]^2 \to [0,1]$ be a function satisfying the following properties:*

$$\begin{aligned} T(a,b) &= T(b,a) \quad \textit{for all } a,b \in [0,1], & (4.1)\\ T(T(a,b),c) &= T(a,T(b,c)) \quad \textit{for all } a,b,c \in [0,1], & (4.2)\\ T(a,b) &\leq T(c,d) \quad \textit{for } a,b,c,d \in [0,1] \textit{ with } a \leq c, b \leq d, & (4.3)\\ T(a,1) &= a \quad \textit{for all } a \in [0,1]. & (4.4) \end{aligned}$$

A function $T : [0,1]^2 \to [0,1]$ that satisfies all these properties is called the triangular norm *or* t-norm. *A t-norm T is called* strictly monotone *if T is a strictly increasing function in the sense that*

$$T(a,b) < T(a',b) \quad \textit{whenever } a, a', b \in (0,1) \textit{ and } a < a'.$$

Triangular norms play an important role also in many-valued logics and in the theory of fuzzy sets. In many-valued logics, they serve as truth degree functions of conjunction connectives. In the fuzzy set theory, they provide a tool for defining various types of the intersection of fuzzy subsets of a given set. For a detailed treatment we refer to the recent book [57].

The axioms (4.1), (4.2), (4.3) and (4.4) are called *commutativity*, *associativity*, *monotonicity* and *boundary condition*, respectively. From the algebraic point of view, a triangular norm is a commutative ordered semigroup with unit element 1 on the unit interval $[0,1]$ of real numbers. Therefore the class of all triangular norms is quite large. Let us consider some important examples.

In connection with a problem on functional equations, Frank [33] introduced the following family $\{T_s \mid s \in (0,\infty),\ s \neq 1\}$ of t-norms:

$$T_s(a,b) = \log_s \left(1 + \frac{(s^a-1)(s^b-1)}{s-1}\right).$$

The limit cases T_M, T_P and T_L defined by

$$\begin{aligned} T_M(a,b) &= \lim_{s\to 0} T_s(a,b) = \min\{a,b\},\\ T_P(a,b) &= \lim_{s\to 1} T_s(a,b) = a\cdot b,\\ T_L(a,b) &= \lim_{s\to\infty} T_s(a,b) = \max\{0, a+b-1\} \end{aligned}$$

are also t-norms. In the literature on many-valued logics, the t-norms T_M, T_P and T_L are often called the *minimum* (or *Gödel*), *product* and *Łukasiewicz* t-norm, respectively. Two other interesting examples are

$$T_F(a,b) = \begin{cases} \min\{a,b\} & \text{if } a+b>1 \\ 0 & \text{otherwise,} \end{cases}$$

introduced by Fodor [31], and the so-called *drastic product*

$$T_D(a,b) = \begin{cases} \min\{a,b\} & \text{if } \max\{a,b\}=1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

It can easily be seen that the minimum t-norm T_M is the maximal t-norm and the drastic product T_D is the minimal t-norm in the pointwise ordering, that is, for every t-norm T,

$$T_D(a,b) \le T(a,b) \le T_M(a,b) \quad \text{whenever } a,b \in [0,1]. \tag{4.6}$$

However, the class of t-norms is not linearly ordered by this pointwise relation. For example, the product t-norm T_P and the Fodor t-norm T_F are not comparable.

A class of functions closely related to the class of t-norms are functions $S : [0,1]^2 \to [0,1]$ such that

$$\begin{aligned} S(a,b) &= S(b,a) \quad \text{for all } a,b \in [0,1], \\ S(S(a,b),c) &= S(a,S(b,c)) \quad \text{for all } a,b,c \in [0,1], \\ S(a,b) &\le S(c,d) \quad \text{for } a,b,c,d \in [0,1] \text{ with } a \le c,\ b \le d, \\ S(a,0) &= a \quad \text{for all } a \in [0,1]. \end{aligned}$$

The functions that satisfy all these properties are called the *triangular conorms* or *t-conorms*.

It can easily be verified, see for example [57], that for each t-norm T, the function $T^* : [0,1]^2 \to [0,1]$ defined for all $a,b \in [0,1]$ by

$$T^*(a,b) = 1 - T(1-a, 1-b) \tag{4.7}$$

is a t-conorm. The converse statement is also true. Namely, if S is a t-conorm, then the function $S^* : [0,1]^2 \to [0,1]$ defined for all $a,b \in [0,1]$ by

$$S^*(a,b) = 1 - S(1-a, 1-b) \tag{4.8}$$

is a t-norm. The t-conorm T^* and t-norm S^*, are called *dual* to the t-norm T and t-conorm S, respectively. For example, the functions S_M, S_P, S_L and S_D defined for $a,b \in [0,1]$ by

$$\begin{aligned}
S_M(a,b) &= \max\{a,b\}, \\
S_P(a,b) &= a+b-a.b, \\
S_L(a,b) &= \min\{1,a+b\}, \\
S_D(a,b) &= \begin{cases} \max\{a,b\} & \text{if } \min\{a,b\}=0, \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}$$

are t-conorms. In the literature, the t-conorms S_M, S_P, S_L and S_D are often called the *maximum, probabilistic sum* , *bounded sum* and *drastic sum*, respectively. It may easily be verified that

$$T_M^* = S_M,\ T_P^* = S_P,\ T_L^* = S_L,\ T_D^* = S_D.$$

The following proposition answers the question whether a triangular norm and triangular conorm are determined uniquely by their values on the diagonal of the unit square. In general, this is not the case, but the extremal t-norms and t-conorms T_M, S_M, T_D, S_D are completely determined by their values on the diagonal of the unit square. For the proof of the following proposition, see [57].

2. Properties of Triangular Norms and Triangular Conorms

PROPOSITION 4.2

(i) *The only t-norm T satisfying $T(x,x) = x$ for all $x \in [0,1]$ is the minimum t-norm T_M.*

(ii) *The only t-conorm S satisfying $S(x,x) = x$ for all $x \in [0,1]$ is the maximum t-conorm S_M.*

(iii) *The only t-norm T satisfying $T(x,x) = 0$ for all $x \in [0,1]$ is the drastic product T_D.*

(iv) *The only t-conorm S satisfying $S(x,x) = 0$ for all $x \in [0,1]$ is the drastic sum S_D.*

The commutativity and associativity properties allow to extend t-norms and t-conorms, introduced as binary operations, to n-ary operations. Let T be a t-norm. We define its extension to more than two arguments by the formula

$$T^{i+1}(a_1, a_2, \ldots, a_{i+2}) = T(T^i(a_1, a_2, \ldots, a_{i+1}), a_{i+2}), \tag{4.9}$$

where $T^1(a_1, a_2) = T(a_1, a_2)$.

For example, the extensions of T_M, T_P, T_L and T_D to m arguments are

$$\begin{aligned}
T_M^{m-1}(a_1, a_2, \ldots, a_m) &= \min\{a_1, a_2, \ldots, a_m\}, \\
T_P^{m-1}(a_1, a_2, \ldots, a_m) &= \prod_{i=1}^{m} a_i, \\
T_L^{m-1}(a_1, a_2, \ldots, a_m) &= \max\{0, \sum_{i=1}^{m} a_i - (m-1)\}, \\
T_D^{m-1}(a_1, a_2, \ldots, a_m) &= \begin{cases} a_i & \text{if } a_j = 1 \text{ for all } j \neq i, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Given a t-conorm S, we can, in a complete analogy, extend this binary operation to m-tuples $(a_1, a_2, \ldots, a_m) \in [0, 1]^m$ by the formula

$$S^{i+1}(a_1, a_2, \ldots, a_{i+2}) = S(S^i(a_1, a_2, \ldots, a_{i+1}), a_{i+2}),$$

where $S^1(a_1, a_2) = S(a_1, a_2)$.

For example, the extensions of S_M, S_P, S_L and S_D to m arguments are

$$\begin{aligned}
S_M^{m-1}(a_1, a_2, \ldots, a_m) &= \max\{a_1, a_2, \ldots, a_m\}, \\
S_P^{m-1}(a_1, a_2, \ldots, a_m) &= 1 - \prod_{i=1}^{m}(1 - a_i), \\
S_L^{m-1}(a_1, a_2, \ldots, a_m) &= \min\{1, \sum_{i=1}^{m} a_i\}, \\
S_D^{m-1}(a_1, a_2, \ldots, a_m) &= \begin{cases} a_i & \text{if } a_j = 0 \text{ for all } j \neq i, \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}$$

If there is no danger of misunderstanding, the upper index $m - 1$ of T or S is omitted.

Now we turn our attention to some algebraic aspects of t-norms and t-conorms, which will be useful later on when T-quasiconcave and T-quasiconvex functions are investigated. Notice that these properties are well-known from the general theory of semigroups.

DEFINITION 4.3 *Let T be a t-norm.*

(i) *An element $a \in [0, 1]$ is called an* idempotent element *of T if $T(a, a) = a$.*

(ii) *An element $a \in (0, 1)$ is called a* nilpotent element *of T if there exists some positive integer $n \in N$ such that $T^{n-1}(a, \ldots, a) = 0$.*

(iii) *An element $a \in (0, 1)$ is called a* zero divisor *of T if there exists some $b \in (0, 1)$ such that $T^{n-1}(a, b) = 0$.*

By definitions and properties of t-norms we obtain the following proposition summarizing the properties of T_M, T_P, T_L and T_D introduced in Definition 4.3.

PROPOSITION 4.4 *Each $a \in [0,1]$ is an idempotent element of T_M. Each $a \in (0,1)$ is both a nilpotent element and zero divisor of T_L, as well as of T_D. The minimum T_M has neither nilpotent elements nor zero divisors, and T_L, as well as T_D possesses only trivial idempotent elements 0 and 1. The product norm T_P has neither non-trivial idempotent elements nor nilpotent elements nor zero divisors.*

Now we add some other usual definitions concerning the properties of triangular norms.

DEFINITION 4.5 *A triangular norm T is said to be*

(i) strict *if it is continuous and strictly monotone,*

(ii) Archimedian *if for all $x, y \in (0,1)$ there exists a positive integer n such that $T^{n-1}(x, \ldots, x) < y$,*

(iii) nilpotent *if it is continuous and if each $a \in (0,1)$ is a nilpotent element of T,*

(iv) idempotent *if each $a \in (0,1)$ is an idempotent element of T.*

Notice that if T is strict, then T is Archimedian. The following proposition summarizes the properties of the most popular t-norms in context to the previous definition; see [57].

PROPOSITION 4.6 *The minimum T_M is neither strict, nor Archimedian, nor nilpotent. The product norm T_P is both strict and Archimedian, but not nilpotent. Łukasiewicz t-norm T_L is both strict and Archimedian and nilpotent. The drastic product T_D is Archimedian and nilpotent, but not continuous, thus not strict.*

Strict monotonicity of t-conorms as well as strict, Archimedian and nilpotent t-conorms can be introduced using the duality (4.7), (4.8). Without presenting all technical details, we only mention that it suffices to interchange the words t-norm and t-conorm and the roles of 0 and 1, respectively, in order to obtain the proper definitions and results for t-conorms.

It is obvious that each strict t-norm T is Archimedian since

$$T(x, x) < T(x, 1) = x$$

for each $x \in (0,1)$. The following result can be found in [114].

PROPOSITION 4.7 *A continuous t-norm T is strict if and only if there exists an automorphism φ of the unit interval $[0,1]$ such that*

$$T(x,y)=\varphi^{-1}(\varphi(x)\varphi(y)).$$

3. Representations of Triangular Norms and Triangular Conorms

Now we show how real-valued functions of one real variable can be used to construct new t-norms and t-conorms, and how new t-norms can be generated from given ones. The construction requires an inverse operation and in order to relax the strong requirement of bijectivity and to replace it by the weaker monotonicity, we first recall some general properties of monotone functions. The following result is crucial for the proper definition of pseudoinverse; see [57].

PROPOSITION 4.8 *Let $f:[a,b]\to[c,d]$ be a non-constant monotone function, where $[a,b]$ and $[c,d]$ are subintervals of the extended real line $\bar{\mathbf{R}}=[-\infty,+\infty]$. Then for each $y\in[c,d]\setminus\mathrm{Ran}(f)$ we have*

$$\begin{aligned}&\sup\{x\in[a,b]\mid(f(x)-y)\cdot(f(b)-f(a))<0\}\\&\qquad=\inf\{x\in[a,b]\mid(f(x)-y)\cdot(f(b)-f(a))>0\}.\end{aligned}$$

This result allows us to introduce the following generalization of an inverse function, where we restrict ourselves to the case of non-constant monotone functions.

DEFINITION 4.9 *Let $[a,b]$ and $[c,d]$ be subintervals of the extended real line $\bar{\mathbf{R}}$. Let $f:[a,b]\to[c,d]$ be a non-constant monotone function. The* pseudo-inverse *$f^{(-1)}:[c,d]\to[a,b]$ is defined by*

$$f^{(-1)}(y)=\sup\{x\in[a,b]\mid(f(x)-y)\cdot(f(b)-f(a))<0\}.$$

The following proposition summarizes some evident consequences of Definition 4.9.

PROPOSITION 4.10 *Let $[a,b]$ and $[c,d]$ be subintervals of the extended real line $\bar{\mathbf{R}}$, and let f be a non-constant function mapping $[a,b]$ into $[c,d]$.*

(i) *If f is non-decreasing, then for all $y\in[c,d]$ we have*

$$f^{(-1)}(y)=\sup\{x\in[a,b]\mid f(x)<y)\}.$$

(ii) *If f is non-increasing, then for all $y\in[c,d]$ we have*

$$f^{(-1)}(y)=\sup\{x\in[a,b]\mid f(x)>y)\}.$$

(iii) *If f is a bijection, then the pseudoinverse $f^{(-1)}$ of f coincides with the inverse function f^{-1} of f.*

(iv) *If f is a strictly increasing function, then the pseudoinverse $f^{(-1)}$ of f is continuous.*

To construct t-norms with the help of functions we start with the best known operations, the usual addition and multiplication of real numbers. The following proposition gives the result known for nearly 200 years and published by N. H. Abel in 1826. We state it here in a bit simplified version without proof.

PROPOSITION 4.11 *Let $f : [a, b] \longrightarrow [c, d]$ be a continuous strictly monotone function, where $[a, b]$ and $[c, d]$ are subintervals of the extended real line $\bar{\mathbf{R}}$. Suppose that $\mathrm{Ran}(f) = [c, d]$ and $\varphi : [c, d] \longrightarrow [a, b]$ is an inverse function to f. Then the function $F : [a, b] \times [a, b] \longrightarrow [a, b]$ defined for every $x, y \in [a, b]$ by*

$$F(x, y) = \varphi(f(x) + f(y)) \tag{4.10}$$

is associative.

If we want to obtain a t-norm by means of (4.10), it is obvious that some additional requirements for f are necessary.

DEFINITION 4.12 *An* additive generator *of a t-norm T is a strictly decreasing function $g : [0, 1] \longrightarrow [0, +\infty]$ which is right continuous at 0, satisfies $g(1) = 0$, and is such that for all $x, y \in [0, 1]$ we have*

$$g(x) + g(y) \in \mathrm{Ran}(g) \cup [g(0), +\infty], \tag{4.11}$$
$$T(x, y) = g^{(-1)}(g(x) + g(y)). \tag{4.12}$$

A multiplicative generator *of a t-norm T is a strictly increasing function $\zeta : [0, 1] \longrightarrow [0, 1]$ which is right continuous at 0, satisfies $\zeta(1) = 1$, and is such that for all $x, y \in [0, 1]$ we have*

$$\zeta(x) \cdot \zeta(y) \in \mathrm{Ran}(\zeta) \cup [0, \zeta(0)], \tag{4.13}$$
$$T(x, y) = \zeta^{(-1)}(\zeta(x).\zeta(y)). \tag{4.14}$$

An additive generator *of a t-conorm S is a strictly increasing function $h : [0, 1] \longrightarrow [0, +\infty]$ which is left continuous at 1, satisfies $h(0) = 0$, and is such that for all $x, y \in [0, 1]$ we have*

$$h(x) + h(y) \in \mathrm{Ran}(g) \cup [h(1), +\infty], \tag{4.15}$$
$$S(x, y) = h^{(-1)}(h(x) + h(y)). \tag{4.16}$$

A multiplicative generator *of a t-conorm S is a strictly decreasing function $\xi : [0,1] \longrightarrow [0,1]$ which is left continuous at 1, satisfies $\xi(0) = 1$, and is such that for all $x, y \in [0,1]$ we have*

$$\xi(x).\xi(y) \in \mathrm{Ran}(\xi) \cup [0, \xi(1)], \tag{4.17}$$

$$S(x,y) = \xi^{(-1)}(\xi(x).\xi(y)). \tag{4.18}$$

Triangular norms (t-conorms) constructed by means of additive (multiplicative) generators are always Archimedian. This property and some other properties of such t-norms are summarized in the following proposition. The corresponding proofs can be found in [57].

PROPOSITION 4.13 *Let $g : [0,1] \longrightarrow [0,+\infty]$ be an additive generator of a t-norm T. Then T is an Archimedian t-norm. Moreover, we have:*

(i) *The t-norm T is strictly monotone if and only if $g(0) = +\infty$.*

(ii) *Each element of $(0,1)$ is a nilpotent element of T if and only if $g(0) < +\infty$.*

(iii) *T is continuous if and only if g is continuous.*

It was mentioned in Example 4.23 that minimum T_M has no nilpotent elements. Since T_M is not strictly monotone it follows from the preceding theorem that it has no additive generator. If a t-norm T is generated by a continuous generator, then by Proposition 4.13, T is an Archimedian t-norm. The following theorem says that the converse statement is also true; see [57] or [32].

PROPOSITION 4.14 *A t-norm T is Archimedian and continuous if and only if there exists a continuous additive generator of T.*

The analogical propositions can be formulated and proved for multiplicative generators and also for t-conorms. The corresponding proofs are left to the reader.

PROPOSITION 4.15 *A t-norm T is Archimedian and continuous if and only if there exists a continuous multiplicative generator ζ of T.*

PROPOSITION 4.16 *A t-conorm S is Archimedian and continuous if and only if there exist both a continuous additive generator h of S, and a multiplicative generator ξ of S.*

EXAMPLE 4.17 *(The Yager t-norms)* One of the most popular families in operations research and particularly in fuzzy linear programming is the family of Yager t-norms; see [135]. The results presented here can be found in [57].

Let $\lambda \in [0, +\infty]$. The Yager t-norm T_λ^Y is defined for all $x, y \in [0,1]$ as follows:

$$T_\lambda^Y(x,y) = \begin{cases} T_D(x,y) & \text{if } \lambda = 0, \\ T_M(x,y) & \text{if } \lambda = +\infty\,, \\ \max\left\{0, 1 - \left((1-x)^\lambda + (1-y)^\lambda\right)^{1/\lambda}\right\} & \text{otherwise.} \end{cases}$$

The Yager t-conorm S_λ^Y is defined for all $x, y \in [0,1]$ as:

$$S_\lambda^Y(x,y) = \begin{cases} S_D(x,y) & \text{if } \lambda = 0, \\ S_M(x,y) & \text{if } \lambda = +\infty, \\ \min\left\{1, \left(x^\lambda + y^\lambda\right)^{1/\lambda}\right\} & \text{otherwise.} \end{cases}$$

(i) Obviously, $T_1^Y = T_L$ and $S_1^Y = S_L$.

(ii) For each $\lambda \in (0, +\infty)$, T_λ^Y and T_λ^Y are dual to each other.

(iii) A Yager t-norm T_λ^Y is nilpotent if and only if $\lambda \in (0, +\infty)$.

(iv) A Yager t-conorm S_λ^Y is nilpotent if and only if $\lambda \in (0, +\infty)$.

(v) If $\lambda \in (0, +\infty)$, then the corresponding continuous additive generator $f_\lambda^Y : [0,1] \to [0,1]$ of the Yager t-norm T_λ^Y is given for all $x \in [0,1]$ by

$$f_\lambda^Y(x) = (1-x)^\lambda\,.$$

(vi) The corresponding continuous additive generator $g_\lambda^Y : [0,1] \to [0,1]$ of the Yager t-conorm S_λ^Y is given for all $x \in [0,1]$ by

$$g_\lambda^Y(x) = x^\lambda.$$

Yager t-norms have been used in several applications of fuzzy set theory. In particular, it was used in extended addition of linear functions with fuzzy parameters. In this context, in [58], it was shown that the sum of piecewise linear function is again piecewise linear, when using Yager t-norm, see also [57]. We shall return to this problem again in Chapter 9. □

4. Negations and De Morgan Triples

Supplementing t-norms and t-conorms by a special unary function we obtain a triplet which is useful in many-valued logics, fuzzy set theory and their applications.

DEFINITION 4.18 *A function $N : [0,1] \to [0,1]$ is called a* negation *if it is non-increasing and satisfies the following conditions:*

$$N(0) = 1, N(1) = 0.$$

Moreover, it is called strict negation, *if it is strictly decreasing and continuous and it is called* strong negation *if it is strict and the following condition of* involution *holds:*

$$N(N(x)) = x \quad \textit{for all } x \in [0,1]. \tag{4.19}$$

Since a strict negation N is a strictly decreasing and continuous function, its inverse N^{-1} is also a strict negation, generally different from N. Obviously, $N^{-1} = N$ if and only if (4.19) holds.

DEFINITION 4.19 *An* intuitionistic negation N_I *is defined as follows*

$$N_I(x) = \begin{cases} 1 & \textit{if } x = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

A weak negation N_W *is defined as follows*

$$N_W(x) = \begin{cases} 1 & \textit{if } x < 1, \\ 0 & \textit{if } x = 1. \end{cases}$$

A standard negation N *is defined by*

$$N(x) = 1 - x. \tag{4.20}$$

Strong negations (including the standard one) defined by

$$N_\lambda(x) = \frac{1-x}{1+\lambda x},$$

where $\lambda > -1$, *are called* λ-complements.

Notice that N_I is not a strict negation, N_W is a dual operation to N_I, i.e., for all $x \in [0,1]$, it holds $N_W(x) = 1 - N_I(1-x)$. The standard negation is a strong negation. An example of strict but not strong negation is the negation N' defined by the formula

$$N'(x) = 1 - x^2.$$

The following proposition characterizing strong negations comes from [32].

PROPOSITION 4.20 *A function* $N : [0,1] \to [0,1]$ *is a strong negation if and only if there exists a strictly increasing continuous surjective function* $\varphi : [0,1] \to [0,1]$ *such that*

$$N(x) = \varphi^{-1}(1 - \varphi(x)).$$

DEFINITION 4.21 *Let T be a t-norm, S be a t-conorm, and N be a strict negation. We say that (T, S, N) is a De Morgan triple if*

$$N(S(x, y)) = T(N(x), N(y)).$$

The following proposition is a simple consequence of the above definitions.

PROPOSITION 4.22 *Let N be a strict negation, T be a t-norm. Let S be defined for all $x, y \in [0, 1]$ as follows:*

$$S(x, y) = N^{-1}(T(N(x), T(y))).$$

Then (T, S, N) is a De Morgan triple. Moreover, if T is continuous, then S is continuous. In addition, if T is Archimedian with an additive generator f, then S is Archimedian with additive generator $g = f \circ N$ and $g(1) = f(0)$.

EXAMPLE 4.23 A Łukasiewicz-like De Morgan triple (T, S, N) is defined as follows:

$$\begin{aligned} T(x, y) &= \varphi^{-1}(\max\{\varphi(x) + \varphi(y) - 1, 0\}), \\ S(x, y) &= \varphi^{-1}(\min\{\varphi(x) + \varphi(y), 1\}), \\ N(x) &= \varphi^{-1}(1 - \varphi(x)), \end{aligned}$$

where $\varphi : [0, 1] \to [0, 1]$ is a strictly increasing continuous surjective function. □

5. Domination of Triangular Norms

Besides the natural pointwise partial ordering of t-norms it is useful to define another relation in the family of t-norms.

DEFINITION 4.24 *A t-norm T' dominates a t-norm T, denoted by $T' \gg T$, if for each $a, b, c, d \in [0, 1]$, the following inequality holds*

$$T'(T(a, b), T(c, d)) \geq T(T'(a, c), T'(b, d)).$$

It is easy to see that if $T' \gg T$, then $T'(x, y) \geq T(x, y)$ for each $x, y \in [0, 1]$, but the opposite in general is not true; see [57]. It can readily be verified that $\gg$ is reflexive and antisymmetric. Moreover, for each t-norm T it holds $T_M \gg T \gg T_D$. The following characterization of domination relation will be useful for further investigation; see [57] for the proof.

PROPOSITION 4.25 *A strict t-norm T_2 with a generator g_2 dominates a strict t-norm T_1 with a generator g_1 if and only if the function $h = g_2 \circ g_1^{-1}$ satisfies the following inequality*

$$h^{-1}(h(x + y) + h(u + v)) \leq h^{-1}(h(x) + h(y)) + h^{-1}(h(u) + h(v))$$

for all nonnegative real numbers x, y, u and v.

6. T-Quasiconcave Functions

As we mentioned in Chapter 3 and will discuss later in Part II of this book, the notions of concavity and convexity of real-valued functions of real variables and its various generalizations have found many applications in economics and engineering.

In contrast to Chapter 3, we now restrict our attention to functions defined on $\mathbf{R}^n$ with range in the unit interval $[0, 1]$ of real numbers. Such functions can be interpreted as membership functions of fuzzy subsets of $\mathbf{R}^n$. We therefore use several terms and some notation of fuzzy set theory. However, it should be pointed out that such functions arise in more contexts. The Greek letter μ, sometimes with an index, will in this chapter denote a function that maps $\mathbf{R}^n$ into the closed unit interval $[0, 1]$ in $\mathbf{R}$. First, several auxiliary notions and some notation will be introduced.

DEFINITION 4.26 *Let μ be a function defined on $\mathbf{R}^n$ with range in $[0, 1]$.*

(i) *The* core *of μ, $\mathrm{Core}(\mu)$, is given by*

$$\mathrm{Core}(\mu) = \{x \in \mathbf{R}^n \mid \mu(x) = 1\}.$$

(ii) *The* support *of μ, $\mathrm{Supp}(\mu)$, is given by*

$$\mathrm{Supp}(\mu) = \mathrm{Cl}(\{x \in \mathbf{R}^n \mid \mu(x) > 0\}),$$

where $\mathrm{Cl}(A)$ is the topological closure of the set $A \subset \mathbf{R}^n$.

(iii) *If $\mathrm{Core}(\mu)$ is nonempty, then μ is said to be* upper-normalized. *If $\mathrm{Core}(1-\mu)$ is nonempty, then μ is said to be* lower-normalized. *Simultaneously upper- and lower-normalized functions are called* normalized.

We have introduced quasiconcave (semi)strictly quasiconcave, quasiconvex and (semi)strictly quasiconvex functions in Definition 3.1. First, we generalize Definition 3.1 by using triangular norms and conorms.

DEFINITION 4.27 *Let X be a nonempty convex subset of $\mathbf{R}^n$, T be a triangular norm, and S be a triangular conorm. A function $\mu : \mathbf{R}^n \to [0, 1]$ is called*

(i) T-quasiconcave *on X if*

$$\mu(\lambda x + (1 - \lambda y)) \geq T(\mu(x), \mu(y)) \tag{4.21}$$

for every $x, y \in X$, $x \neq y$ and $\lambda \in (0, 1)$;

(ii) strictly T-quasiconcave *on X if*

$$\mu(\lambda x + (1 - \lambda)y) > T(\mu(x), \mu(y)) \tag{4.22}$$

for every $x, y \in X$, $x \neq y$ *and* $\lambda \in (0, 1)$*;*

(iii) semistrictly T-quasiconcave *on* X *if (4.21) holds for every* $x, y \in X$, $x \neq y$ *and* $\lambda \in (0, 1)$ *and (4.22) holds for every* $x, y \in X$ *and* $\lambda \in (0, 1)$ *such that* $\mu(x) \neq \mu(y)$*;*

(iv) S-quasiconvex *on* X *if*

$$\mu(\lambda x + (1 - \lambda y)) \leq S(\mu(x), \mu(y)) \tag{4.23}$$

for every $x, y \in X$, $x \neq y$ *and* $\lambda \in (0, 1)$*;*

(v) strictly S-quasiconvex *on* X *if*

$$\mu(\lambda x + (1 - \lambda)y) < S(\mu(x), \mu(y)) \tag{4.24}$$

for every $x, y \in X$, $x \neq y$ *and* $\lambda \in (0, 1)$*;*

(vi) semistrictly S-quasiconvex *on* X *if (4.23) holds for every* $x, y \in X$, $x \neq y$ *and* $\lambda \in (0, 1)$ *and (4.24) holds for every* $x, y \in X$ *and* $\lambda \in (0, 1)$ *such that* $\mu(x) \neq \mu(y)$*;*

(vii) (strictly, semistrictly) (T, S)-quasimonotone *on* X *if* μ *is (strictly, semistrictly)* T*-quasiconcave and (strictly)* S*-quasiconvex on* X*, respectively;*

(viii) (strictly, semistrictly) T-quasimonotone *on* X *if* μ *is (strictly, semistrictly)* T*-quasiconcave and (strictly, semistrictly)* T^**-quasiconvex on* X*, where* T^* *is the dual t-conorm to* T*.*

Later on, we further generalize Definition 4.27 adapting the concept of (Φ, Ψ)-concave functions introduced in Section 3.3.

Obviously, the class of quasiconcave functions that map $\mathbf{R}^n$ into $[0, 1]$ is exactly the class of T_M-quasiconcave functions and the class of quasiconvex functions that map $\mathbf{R}^n$ into $[0, 1]$ is exactly the class of S_M-quasiconvex functions.

Similarly, the class of quasimonotone functions that map $\mathbf{R}^n$ into $[0, 1]$ is exactly the class of (T_M, S_M)-quasimonotone functions. As $S_M = T_M^*$, we have, by (viii) in Definition 4.27, that this is the class of T_M-quasimonotone functions. Moreover, since the minimum triangular norm T_M is the maximal t-norm, and the drastic product T_D is the minimal t-norm, we have the following consequence of Definition 4.27.

PROPOSITION 4.28 *Let* X *be a nonempty convex subset of* $\mathbf{R}^n$*,* μ *be a function,* $\mu : \mathbf{R}^n \to [0, 1]$*, and* T *be a triangular norm.*

(i) *If* μ *is (strictly, semistrictly) quasiconcave on* X*, then* μ *is (strictly, semistrictly)* T*-quasiconcave on* X *, respectively.*

(ii) *If μ is (strictly, semistrictly) T-quasiconcave on X, then μ is (strictly, semistrictly) T_D-quasiconcave on X, respectively.*

Analogously, the maximum triangular conorm S_M is the minimal conorm and the drastic sum S_D is the maximal conorm, hence Proposition 4.28 can be reformulated for S-quasiconvex functions.

It is easy to show that there exist T-quasiconcave functions that are not quasiconcave (see Example 4.33), and there exist strictly or semistrictly T-quasiconcave functions that are not strictly or semistrictly quasiconcave. Nevertheless, in the one-dimensional Euclidean space $\mathbf{R}$, the following proposition is of some interest.

PROPOSITION 4.29 *Let X be a nonempty convex subset of $\mathbf{R}$, let T be a triangular norm, and let $\mu : \mathbf{R} \to [0,1]$ be such that $\mu(\bar{x}) = 1$ for some $\bar{x} \in X$. If μ is (strictly, semistrictly) T-quasiconcave on X, then μ is (strictly, semistrictly) quasiconcave on X.*

PROOF. We prove only the part concerning T-quasiconcavity. The part concerning strict (semistrict) T-quasiconcavity can be verified analogously. Let λ be in $[0,1]$ and let x and y in X. Without loss of generality we assume that $x \leq y$. Thus $x \leq \lambda x + (1-\lambda)y$. If $\lambda x + (1-\lambda)y \leq \bar{x}$, then there exists $\alpha \in [0,1]$ such that

$$\lambda x + (1-\lambda)y = \alpha x + (1-\alpha)\bar{x}.$$

By T-quasiconcavity of μ on X and properties of triangular norms, we have

$$\mu(\alpha x + (1-\alpha)\bar{x}) \geq T(\mu(x), \mu(\bar{x})) = T(\mu(x), 1) = \mu(x).$$

Since $\mu(\lambda x + (1-\lambda)y) = \mu(\alpha x + (1-\alpha)\bar{x})$ and $\mu(x) \geq \min\{\mu(x), \mu(y)\}$, we have

$$\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}.$$

It remains to show that this inequality holds also in the case that $\bar{x} < \lambda x + (1-\lambda)y$. Since $\lambda x + (1-\lambda)y \leq y$, we can use an analogous argument. There exists $\beta \in [0,1]$ such that $\lambda x + (1-\lambda)y = \beta\bar{x} + (1-\beta)y$ and consequently

$$\begin{aligned}
\mu(\lambda x + (1-\lambda)y) &= \mu(\beta\bar{x} + (1-\beta)y) \geq T(\mu(\bar{x}), \mu(y)) \\
&= T(1, \mu(y)) = T(\mu(y), 1) = \mu(y) \\
&\geq \min\{\mu(x), \mu(y)\}.
\end{aligned}$$

■

Analogous propositions are valid for S-quasiconvex functions and for T-quasimonotone functions.

PROPOSITION 4.30 *Let X be a nonempty convex subset of* $\mathbf{R}$, *let S be a triangular conorm, and let $\mu : \mathbf{R} \to [0,1]$ be such that $\mu(\hat{x}) = 0$ for some $\hat{x} \in X$. If μ is (strictly, semistrictly) S-quasiconvex on X, then μ is (strictly, semistrictly) quasiconvex on X.*

To prove Proposition 4.30 we can use the following relationship between T-quasiconcave and S-quasiconvex functions.

PROPOSITION 4.31 *Let X be a nonempty convex subset of $\mathbf{R}^n$, let T be a triangular norm and let $\mu : \mathbf{R}^n \to [0,1]$ be (strictly, semistrictly) T-quasiconcave on X. Then $\mu^* = 1 - \mu$ is (strictly, semistrictly) T^*-quasiconvex on X, where T^* is the t-conorm dual to T.*

PROOF. The proof follows directly from Definition 4.27 and relation (4.7). ■

The following proposition is a consequence of Propositions 4.29 and 4.30.

PROPOSITION 4.32 *Let X be a nonempty convex subset of* $\mathbf{R}$, *let T and S be a t-norm and t-conorm, respectively, and let $\mu : \mathbf{R} \to [0,1]$ be normalized. If μ is (strictly, semistrictly) (T,S)-quasimonotone on X, then μ is (strictly, semistrictly) quasimonotone on X.*

In what follows we shall use the above relationship between T-quasiconcave and T^*-quasiconvex functions restricting ourselves only to T-quasiconcave functions. Usually, with some exceptions, the dual formulation for S-quasiconvex functions will not be explicitly mentioned. The following example shows that the assumption of (upper, lower)-normality of μ is essential for the validity of Propositions 4.29 and 4.30.

EXAMPLE 4.33 Let $X = [-\pi, \pi]$, and let $\mu : \mathbf{R} \to [0,1]$ be defined by

$$\mu(x) = \begin{cases} \frac{1}{6}\sin x + \frac{1}{2} & \text{if } x \in X, \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

see Figure 4.1. We show that μ is T_P-quasiconcave, T_P^*-quasiconvex, i.e., T_P-quasimonotone on X (and also on $\mathbf{R}$). From Figure 4.1 it is seen that μ is neither quasiconcave nor quasiconvex on X (and also on $\mathbf{R}$). Let us show that μ is T_P-quasiconcave for the product norm $T_P(x,y) = xy$. Observe that $1/3 \le \mu(z) \le 2/3$ for every real z. Evidently, we can restrict our considerations only to interval X. Let $x, y \in X$ be such that $x \le y$, let $\lambda \in [0,1]$ and put $z = \lambda x + (1-\lambda)y$. Then $x \le z \le y$. Suppose that $z \in [-\pi, 0)$. Then $\mu(z) \ge 1/3$,

$$\max\{\mu(x) \mid x \le z\} = \mu(-\pi) = \frac{1}{2}$$

and

$$\max\{\mu(y) \mid z \leq y\} = \mu(\frac{\pi}{2}) = \frac{2}{3}.$$

Thus

$$\mu(z) \geq \frac{1}{3} = \frac{1}{2} \cdot \frac{2}{3} \geq \mu(x) \cdot \mu(y).$$

Now, suppose that $z \in [0, \pi]$. Then $\mu(z) \geq 1/2$,

$$\max\{\mu(y) \mid z \leq y\} = \mu(\frac{\pi}{2}) = \frac{2}{3}$$

and

$$\max\{\mu(y) \mid z \leq y\} = \mu(\frac{\pi}{2}) = \frac{2}{3}.$$

Thus

$$\mu(z) \geq \frac{1}{2} > \frac{2}{3} \cdot \frac{2}{3} \geq \mu(x) \cdot \mu(y).$$

We have shown that

$$\mu(\lambda x + (1 - \lambda)y) \geq T_P(\mu(x), \mu(y)),$$

i.e., μ is T_P-quasiconcave on X (and also on $\mathbf{R}$). To demonstrate that μ is T_P^*-quasiconvex on X (and also T_P-quasimonotone on X), observe that $1 - \mu$ is T_P-quasiconcave on X (and also on $\mathbf{R}$). Applying Proposition 4.31 (duality between t-norms and t-conorms) we get the desired result. □

In the sequel we return to the problem formulated at the end of the previous chapter, particularly, under what condition the intersection $\bigcap \mathrm{Ker}\,(U(\mu, \alpha))$ is nonempty.

The following theorem is a counterpart of Theorem 3.11.

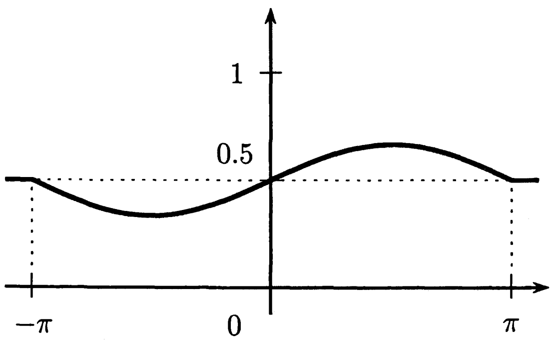

Figure 4.1.

THEOREM 4.34 *Let X be a nonempty convex subset of $\mathbf{R}^n$, let $\mu : \mathbf{R}^n \to [0,1]$ be upper-normalized. Then the following are equivalent:*

(i)

$$\mathrm{Core}(\mu) \subset \bigcap_{\alpha\in[0,1]} \mathrm{Ker}\,(U(\mu,\alpha)). \tag{4.25}$$

(ii) *μ is quasiconcave from $\bar{x}$ for every $\bar{x} \in$ Core(μ).*

(iii) *μ is T_D-quasiconcave on X.*

PROOF. (i) $\Rightarrow$ (ii): To prove this, observe that it is sufficient to show that for each $x \in X$ and $\bar{x} \in \mathrm{Core}(\mu)$, function $\mu_{x,\bar{x}}$ is nondecreasing on $[0,1]$. Let $x \in X$, $y = x + \lambda(\bar{x} - x)$ for an arbitrary $\lambda \in [0,1]$ and set $\beta = \mu(x) = \mu_{x,\bar{x}}(0)$. Then $x \in U(\mu,\beta)$ and by (4.25) we have $\bar{x} \in \mathrm{Ker}\,(U(\mu,\beta))$. As μ is supposed to be upper-starshaped on X, it follows that upper-level set $U(\mu,\beta)$ is starshaped, thus $y = x + \lambda(\bar{x} - x) \in U(\mu,\beta)$. Hence, $\mu(y) = \mu_{x,\bar{x}}(\lambda) \geq \beta = \mu_{x,\bar{x}}(0)$. From the last inequality we obtain that $\mu_{x,\bar{x}}$ is nondecreasing on $[0,1]$.

(ii) $\Rightarrow$ (iii): Suppose that $\mu_{x,\bar{x}}$ is quasiconcave on $\mathbb{L}_X(x,\bar{x})$ for every $x \in X$ and every $\bar{x} \in \mathrm{Core}(\mu)$. Let $x', x \in X$. First, suppose $\mu(x') = 1$, then $x' \in \mathrm{Core}(\mu)$ and, by assumption, $\mu_{x,x'}$ is quasiconcave on $\mathbb{L}_X(x,x')$. Setting $y = \lambda x + (1-\lambda)x'$ for an arbitrary $\lambda \in [0,1]$, it follows that $\mu(y) \geq T_M(\mu(x'),\mu(x))$. Taking into account that $T_M \geq T_D$, we obtain the required result $\mu(y) \geq T_D(\mu(x'),\mu(x))$.

Second, assume $\mu(x') < 1$, $\mu(x) < 1$, then by definition (4.5) we have $T_D(\mu(x'),\mu(x)) = 0$ and, again, setting $y = \lambda x + (1-\lambda)x'$ for an arbitrary $\lambda \in [0,1]$, we get $\mu(y) \geq T_D(\mu(\bar{x}),\mu(x))$. Consequently, μ is T_D-quasiconcave on X.

(iii) $\Rightarrow$ (i): Since μ is upper-normalized, there exists $\bar{x} \in \mathrm{Core}(\mu)$. To prove (4.25) we will show that $\bar{x} \in \bigcap_{\alpha\in[0,1]} \mathrm{Ker}\,(U(\mu,\alpha))$. Let $\alpha \in [0,1]$, it follows that $\bar{x} \in U(\mu,\alpha)$. Suppose that $\bar{x} \notin \mathrm{Ker}\,(U(\mu,\alpha))$. Then there exists $x \in U(\mu,\alpha)$, $x \neq \bar{x}$ and $\lambda \in (0,1)$, such that $x + \lambda(\bar{x} - x) \notin U(\mu,\alpha)$. By T_D-quasiconcavity of μ, we obtain successively $\mu(x + \lambda(\bar{x} - x)) \geq T_D(\mu(x),\mu(\bar{x})) = T_D(\mu(x),1) = \mu(x)$. Since $x \in U(\mu,\alpha)$, we get $\mu(x) \geq \alpha$, thus $x + \lambda(\bar{x} - x) \in U(\mu,\alpha)$, a contradiction. Consequently, $\bar{x} \in \mathrm{Ker}\,(U(\mu,\alpha))$ for every $\alpha \in [0,1]$. ■

COROLLARY 4.35 *If μ is upper-normalized and T-quasiconcave on X, where T is a t-norm, then μ is upper-starshaped on X.*

PROOF. As T_D is the minimal t-norm, i.e., $T \geq T_D$ for each t-norm T, it follows that μ is T_D-quasiconcave on X. Hence, by the preceding theorem, (4.25) holds and this condition implies upper-starshapedness of μ. ■

The equivalence of (i) and (ii) in the preceding theorem makes it clear that an upper-normalized membership function is T_D-quasiconcave if and only if it is quasiconcave on each line going through the points with the value 1. The following theorem is obtained by reformulating Theorem 4.34 in terms of "dual notions". The proof, which may be performed in a completely analogous way, is left to the reader.

THEOREM 4.36 *Let X be a nonempty convex subset of $\mathbf{R}^n$, let $\mu : \mathbf{R}^n \to [0, 1]$ be lower-normalized. Then the following are equivalent:*

(i)

$$\mathrm{Core}(1 - \mu) \subset \bigcap_{\alpha \in [0,1]} \mathrm{Ker}\,(L(\mu, \alpha)).$$

(ii) *μ is quasiconvex from $\bar{x}$ for every $\bar{x} \in$ Core$(1 - \mu)$.*

(iii) *μ is S_D-quasiconvex on X.*

EXAMPLE 4.37 Let $\mu : \mathbf{R}^2 \to [0, 1]$ be defined as follows:

$$\mu(x, y) = \begin{cases} 1 & \text{if } x \in X_2, \\ 0 & \text{otherwise,} \end{cases}$$

where X_2 is the "moon-shaped" set from Example 2.7, see Figure 2.1 (b). Clearly, function μ has the following properties:

- μ is upper-normalized, Core$(\mu) = X_2$.
- $\bigcap_{\alpha \in I_\mu} \mathrm{Ker}\,(U(\mu, \alpha)) = \mathrm{Ker}(X_2)$.
- Condition (4.25) is violated.
- μ is upper-starshaped on $\mathbf{R}^2$.
- μ is not quasiconcave from x, where x is one of the end points of the "moon-shaped" set X_2.
- μ is not T_D-quasiconcave on $\mathbf{R}^2$.

□

Now, we formulate an analogue to Theorem 4.34 or Theorem 4.36 for T-quasimonotone functions. The proof is a straightforward analogue of the proof of Theorem 4.34 and, therefore, it is omitted.

THEOREM 4.38 *Let X be a convex subset of $\mathbf{R}^n$, let $\mu : \mathbf{R}^n \to [0,1]$ be normalized. Then the following are equivalent.*

(i)

$$\mathrm{Core}(\mu) \subset \bigcap_{\alpha \in [0,1]} \mathrm{Ker}\left(U(\mu,\alpha)\right), \mathrm{Core}(1-\mu) \subset \bigcap_{\alpha \in [0,1]} \mathrm{Ker}\left(L(\mu,\alpha)\right).$$

(ii) *μ is quasiconcave from $\bar{x}$ for every $\bar{x} \in$ Core(μ), and quasiconvex from $\hat{x}$ for every $\hat{x} \in$ Core$(1-\mu)$.*

(iii) *μ is T_D-quasimonotone on X.*

Up to now we have investigated the class of T-quasiconcave functions based on general t-norms. Now, we look closer at a narrower class of T-quasiconcave functions based on a t-norm T generated by a generator function g, see Definition 4.12.

PROPOSITION 4.39 *Let X be a nonempty convex subset of $\mathbf{R}^n$, $g : [0,1] \to [0,+\infty]$ be an additive generator of a t-norm T, and $\mu : \mathbf{R}^n \to [0,1]$. Then μ is T-quasiconcave on X if and only if the following inequality holds*

$$g(\mu(\lambda x + (1-\lambda)y) \leq g(\mu(x)) + g(\mu(y)) \tag{4.26}$$

for each $x, y \in X$ and every $\lambda \in [0,1]$.

PROOF. Let $x, y \in X$ and $\lambda \in [0,1]$. If μ is T-quasiconcave on X, then by (4.12) we obtain $\mu(\lambda x + (1-\lambda)y \geq T(\mu(x), \mu(y)) = g^{(-1)}(g(\mu(x)) + g(\mu(y)))$. As g is decreasing on $[0,1]$, we immediately obtain $g(\mu(\lambda x + (1-\lambda)y) \leq g(g^{(-1)}(g(\mu(x)) + g(\mu(y))))$, which gives the required result (4.26).

Conversely, let (4.26) hold and let $x, y \in X$, $\lambda \in [0,1]$. Observe that $g^{(-1)}$ is nonincreasing on $[0,+\infty]$. From this fact and from (4.12), (4.26), it follows that $\mu(\lambda x + (1-\lambda)y) = g^{(-1)}(g(\mu(\lambda x + (1-\lambda)y)) \geq g^{(-1)}(g(\mu(x)) + g(\mu(y))) = T(\mu(x), \mu(y))$. Consequently, μ is T-quasiconcave on X. ∎

COROLLARY 4.40 *If $g \circ \mu$ is quasiconvex on X, then μ is T-quasiconcave on X.*

PROOF. Observe that the following inequality

$$\max\{g(\mu(x)), g(\mu(y)\} \leq g(\mu(x)) + g(\mu(y)) \tag{4.27}$$

is valid for every $x, y \in X$. If $g \circ \mu$ is quasiconvex on X, then, by inequality (3.5) in Definition 3.1, we obtain

$$g(\mu(\lambda x + (1-\lambda)y) \leq \max\{g(\mu(x)), g(\mu(y)\}.$$

Combining this inequality with (4.27) we obtain (4.26) and by Proposition 4.39 the required result follows immediately. ■

In order to formulate an analogue for T-quasiconvex functions, we have to be careful about the inequality relation in the counterpart of (4.26).

PROPOSITION 4.41 *Let X be a nonempty convex subset of $\mathbf{R}^n$, $h : [0,1] \to [0,+\infty]$ be an additive generator of a t-conorm S, and $v : \mathbf{R}^n \to [0,1]$ be a function. Then v is S-quasiconvex on X if and only if the following inequality holds*

$$h(v(\lambda x + (1-\lambda)y) \le h(v(x)) + h(v(y)) \tag{4.28}$$

for each $x, y \in X$ and every $\lambda \in [0,1]$.

PROOF. Let $x, y \in X$ and $\lambda \in [0,1]$. If v is S-quasiconvex on X, then by analogy to (4.12) we obtain

$$\begin{aligned} v(\lambda x + (1-\lambda)y &\le S(v(x), v(y)) \\ &= h^{(-1)}(h(v(x)) + h(v(y))). \end{aligned}$$

As h is increasing on $[0,1]$, we immediately obtain

$$h(v(\lambda x + (1-\lambda)y) \le h(h^{(-1)}(h(v(x)) + h(v(y)))),$$

which gives the required result (4.28).

Conversely, let (4.28) hold and let $x, y \in X$, $\lambda \in [0,1]$. Observe that $h^{(-1)}$ is nondecreasing on $[0,+\infty]$. From this fact and from (4.12), (4.28), it follows that

$$\begin{aligned} v(\lambda x + (1-\lambda)y) &= h^{(-1)}(h(v(\lambda x + (1-\lambda)y)) \\ &\le h^{(-1)}(h(v(x)) + h(v(y))) = S(v(x), v(y)). \end{aligned}$$

Consequently, v is S-quasiconvex on X. ■

COROLLARY 4.42 *If $h \circ v$ is quasiconvex on X, then v is S-quasiconvex on X.*

It may be of some interest to combine the preceding Propositions to obtain analogous results for T-quasimonotone functions, i.e., (T, T^*)-quasimonotone functions, see (vi) in Definition 4.27. The proofs of the following proposition and corollary follow directly from the two preceding propositions and, therefore, they are omitted.

PROPOSITION 4.43 *Let X be a nonempty convex subset of $\mathbf{R}^n$ and let $g : [0,1] \to [0,+\infty]$ be an additive generator of a t-norm T. Then a function $\mu : \mathbf{R}^n \to [0,1]$ is T-quasimonotone on X if and only if the following inequalities hold*

$$g(\mu(x)) + g(\mu(y)) - 1 \leq g(\mu(\lambda x + (1-\lambda)y) \leq g(\mu(x)) + g(\mu(y))$$

for each $x, y \in X$ and every $\lambda \in [0,1]$.

COROLLARY 4.44 *If $g \circ \mu$ is quasimonotone on X, then μ is T-quasimonotone on X.*

In the following proposition we show how a new T-quasiconcave function may be created by two T-quasiconcave functions, provided that they are aggregated by a t-norm T' that dominates T. An extension to more than two membership functions makes no problem.

PROPOSITION 4.45 *Let X be a nonempty convex subset of $\mathbf{R}^n$, let T and T' be t-norms and let $\mu_i : \mathbf{R}^n \to [0,1], i = 1,2,$ be T-quasiconcave on X. If T' dominates T, then $\varphi : \mathbf{R}^n \to [0,1]$ defined by*

$$\varphi(x) = T'(\mu_1(x), \mu_2(x)), \quad x \in \mathbf{R}^n,$$

is T-quasiconcave on X.

PROOF. As $\mu_i, i = 1,2,$ are T-quasiconcave on X, we have

$$\mu_i(\lambda x + (1-\lambda)y) \geq T(\mu_i(x), \mu_i(y))$$

for every $\lambda \in [0,1]$ and $x, y \in X$. By monotonicity of T', we obtain

$$\begin{aligned} \varphi(\lambda x + (1-\lambda)y) &= T'(\mu_1(\lambda x + (1-\lambda)y), \mu_2(\lambda x + (1-\lambda)y)) \\ &\geq T'(T(\mu_1(x), \mu_1(y)), T(\mu_2(x), \mu_2(y))). \end{aligned} \tag{4.29}$$

Using the fact that T' dominates T, we obtain

$$\begin{aligned} &T'(T(\mu_1(x), \mu_1(y)), T(\mu_2(x), \mu_2(y))) \\ &\quad \geq T(T'(\mu_1(x), \mu_2(x)), T'(\mu_1(y), \mu_2(y))) = T(\varphi(x), \varphi(y)). \end{aligned} \tag{4.30}$$

Combining (4.29) and (4.30) we obtain the required result. ■

COROLLARY 4.46 *Let X be a convex subset of $\mathbf{R}^n$, let T be a t-norm, and let $\mu_i : \mathbf{R}^n \to [0,1], i = 1,2,$ be T-quasiconcave on X. Then $\varphi_i : \mathbf{R}^n \to [0,1]$, $i = 1,2,$ defined by*

$$\begin{aligned} \varphi_1(x) &= T(\mu_1(x), \mu_2(x)), \quad x \in \mathbf{R}^n, \\ \varphi_2(x) &= T_M(\mu_1(x), \mu_2(x)), \quad x \in \mathbf{R}^n, \end{aligned}$$

are also T-quasiconcave on X.

PROOF. The proof follows from the preceding proposition and the evident fact that T dominates T and T_M dominates every t-norm T. ■

7. (Φ, T)-Concave Functions

Now, we further generalize Definition 4.27 adapting the concept of (Φ, Ψ)-concave functions introduced in Section 3.3.

DEFINITION 4.47 *Let X be a nonempty subset of $\mathbf{R}^n$, Φ be a set of mappings $\varphi : X \times X \times [0,1] \to \mathbf{R}^n$, $T : [0,1]^2 \to [0,1]$ be a t-norm, S be a t-conorm. Let X be a Φ-convex subset of $\mathbf{R}^n$. A function $\mu : \mathbf{R}^n \to [0,1]$ is called*

(i) (Φ, T)-concave on X *if for each $x, y \in X$ there exists $\varphi \in \Phi$ such that for every $\lambda \in [0,1]$*
$$\varphi(x, y, \lambda) \in X$$
and
$$\mu(\varphi(x, y, \lambda)) \geq T(\mu(x), \mu(y));$$

(ii) strictly (Φ, T)-concave on X *if for each $x, y \in X$ there exists $\varphi \in \Phi$ such that for every $\lambda \in (0,1)$*
$$\varphi(x, y, \lambda) \in X$$
and
$$\mu(\varphi(x, y, \lambda)) > T(\mu(x), \mu(y));$$

(iii) semistrictly (Φ, T)-concave on X *if for each $x, y \in X$ there exists $\varphi \in \Phi$ such that for every $\lambda \in [0,1]$*
$$\varphi(x, y, \lambda) \in X,$$
$$\mu(\varphi(x, y, \lambda)) \geq T(\mu(x), \mu(y)),$$
and
$$\mu(\varphi(x, y, \lambda)) > T(\mu(x), \mu(y))$$
whenever $\mu(x) \neq \mu(y)$.

(iv) (Φ, S)-convex on X *if for each $x, y \in X$ there exists $\varphi \in \Phi$ such that for every $\lambda \in [0,1]$*
$$\varphi(x, y, \lambda) \in X$$
and
$$\mu(\varphi(x, y, \lambda)) \leq S(\mu(x), \mu(y));$$

(v) strictly (Φ, S)-convex on X *if for each $x, y \in X$ there exists $\varphi \in \Phi$ such that for every $\lambda \in (0,1)$*
$$\varphi(x, y, \lambda) \in X$$
and
$$f(\varphi(x, y, \lambda)) < S(\mu(x), \mu(y));$$

(vi) semistrictly (Φ, S)-convex on X *if for each* $x, y \in X$ *there exists* $\varphi \in \Phi$ *such that for every* $\lambda \in (0, 1)$

$$\varphi(x, y, \lambda) \in X,$$

$$\mu(\varphi(x, y, \lambda)) \leq S(\mu(x), \mu(y)),$$

and

$$\mu(\varphi(x, y, \lambda)) < S(\mu(x), \mu(y))$$

whenever $\mu(x) \neq \mu(y)$.

(vii) (Φ, T, S)-(strictly, semistrictly) monotone on X *if it is both* (Φ, S)-*(strictly, semistrictly) convex on* X *and* (Φ, T)-*(strictly, semistrictly) concave on* X*;*

(viii) (Φ, T)-(strictly, semistrictly) monotone on X *if* μ *is (strictly, semistrictly)* (Φ, T)-*concave and (strictly, semistrictly)* (Φ, T^*)-*convex on* X*, where* T^* *is a dual t-conorm to* T*;*

(ix) (Φ, S)-(strictly, semistrictly) monotone on X *if* μ *is (strictly, semistrictly)* (Φ, S)-*convex and (strictly, semistrictly)* (Φ, S^*)-*concave on* X*, where* S^* *is the dual t-norm to* S.

Notice that each (Φ, T)-concave function from Definition 4.47 is (Φ, Ψ)-concave by Definition 3.21, where Ψ contains a single function, i.e., $\Psi = \{\psi\}$, with

$$\psi(x, y, \alpha, \beta, \lambda) = T(\alpha, \beta) \tag{4.31}$$

for all $x, y \in X$, $\alpha, \beta, \lambda \in [0, 1]$. Analogically, any (Φ, S)-convex function from Definition 4.47 is (Φ, Ψ)-convex by Definition 3.21, where Ψ contains a single function, i.e., $\Psi = \{\psi\}$, with

$$\psi(x, y, \alpha, \beta, \lambda) = S(\alpha, \beta)$$

for all $x, y \in X$, $\alpha, \beta, \lambda \in [0, 1]$.

Clearly, if Φ and Ψ are singletons, i.e., $\Phi = \{\varphi\}$ and $\Psi = \{\psi\}$, where

$$\varphi(x, y, \lambda) = x + \lambda(y - x)$$

and $\psi(x, y, \alpha, \beta, \lambda)$ is defined by (4.31) for all $x, y \in X$, $\alpha, \beta \in \mathbf{R}$, $\lambda \in [0, 1]$, then by Definition 4.47 we obtain (strictly, semistrictly) quasiconcave functions on the convex set X defined already by Definition 4.27.

Let Φ consists of all mappings $\varphi : X \times X \times [0, 1] \to \mathbf{R}^n$ with $\varphi(x, y, \cdot)$ being continuous on $[0, 1]$ and such that for each $x, y \in X$, $\varphi(x, y, 0) = x$ and $\varphi(x, y, 1) = y$. Let Ψ be as in (4.31), i.e., $\Psi = \{\psi\}$. Then by Definition 4.47 we obtain generalized (strictly, semistrictly) UQCN and LQCN functions on the path-connected set X.

8. Properties of (Φ, T)-Concave Functions

In this section we derive some important properties of (Φ, T)-concave functions being based on a triangular norm T. As we demonstrated in the previous section, (Φ, T)-concave functions are nothing else than (Φ, Ψ)-concave functions for the particular $\Psi = \{\psi\}$, where ψ is defined by (4.31). Therefore all results derived in Chapter 3, Section 3.2, are applicable.

The first problem is whether all upper level sets of a (Φ, T)-concave function are Φ-convex. Trying to apply Proposition 3.23, we find out that (3.20) is satisfied only if $T = T_M$, otherwise Φ-convexity of upper level sets is not secured by this proposition. It is easy to find an upper starshaped function which is not quasiconcave, hence some of its upper level sets are not convex. Such a function demonstrates that upper level sets of (Φ, T)-concave functions are not necessarily Φ-convex. On the other hand, the hypograph a (Φ, T)-concave function is in a sense Φ-convex. We have the following proposition.

PROPOSITION 4.48 *Let X be a nonempty Φ-convex subset of $\mathbf{R}^n$, Φ be a given set of mappings $\varphi : X \times X \times [0,1] \to \mathbf{R}^n$, and $T : [0,1]^2 \to [0,1]$ be a t-norm. If $\mu : \mathbf{R}^n \to [0,1]$ is (Φ, T)-concave on X, then the hypograph of μ is a $\bar{\Phi}$-convex subset of $\mathbf{R}^{n+1}$, where $\bar{\Phi} = \{\bar{\varphi} \mid \bar{\varphi} = (\varphi, \psi), \varphi \in \Phi\}$ and ψ is defined by (4.31).*

PROOF. Let $(x, \alpha), (y, \beta) \in \mathrm{Hyp}(\mu)$, $\lambda \in [0,1]$. Then

$$\mu(x) \geq \alpha, \quad \mu(y) \geq \beta. \tag{4.32}$$

We have to show that there exists a $\bar{\varphi} \in \bar{\Phi}$, such that $\bar{\varphi}((x, \alpha), (y, \beta), \lambda) \in \mathrm{Hyp}(\mu)$.

Since X is Φ-convex subset of $\mathbf{R}^n$ and $x, y \in X$, there exists $\varphi \in \Phi$ such that $\varphi(x, y, \lambda) \in X$, and by (Φ, T)-concavity of μ on X, we get

$$\mu(\varphi(x, y, \lambda)) \geq T(\mu(x), \mu(y)). \tag{4.33}$$

By (4.32) and (4.33) and monotonicity of T, we obtain

$$f(\varphi(x, y, \lambda)) \geq T(\alpha, \beta).$$

Setting $\bar{\varphi} = (\varphi, \psi)$, where $\psi(x, y, \alpha, \beta, \lambda) = T(\alpha, \beta)$, we finally obtain

$$\bar{\varphi}((x, \alpha), (y, \beta), \lambda) \in \mathrm{Hyp}(\mu).$$

■

The second problem is whether some favorable local - global properties, similar to those in Theorem 3.25 and 3.26, are valid for (Φ, T)-concave functions. Unfortunately, the crucial assumption of those theorems, namely, inequality (3.20), is satisfied only if $T = T_M$. We can, however, derive the same

results for normalized (Φ, T)-concave functions independently of this assumption.

THEOREM 4.49 *Let X be a nonempty Φ-convex subset of $\mathbf{R}^n$, where Φ be a set of mappings $\varphi : X \times X \times [0,1] \to \mathbf{R}^n$ such that, for all $x, y \in X$,*

$$\lim_{\lambda \to 0_+} \varphi(x, y, \lambda) = x. \tag{4.34}$$

Moreover, let $T : [0,1]^2 \to [0,1]$ be a t-norm, $\mu : \mathbf{R}^n \to [0,1]$ be (Φ, T)-concave on X, and $\mathrm{Core}(\mu) \cap X$ *be nonempty. If $\bar{x} \in X$ is a strict local maximizer of μ on X, then it is a strict global maximizer of μ on X.*

PROOF. Suppose that $\bar{x} \in X$ is not a global maximizer. Then there exists $y \in X$, such that

$$\mu(\bar{x}) < \mu(y).$$

Moreover, we have $y \in \mathrm{Core}(\mu)$. Since $\bar{x} \in X$ is a strict local maximizer, then there exists an open ball B with center at $\bar{x} \in X$, such that

$$\mu(x) < \mu(\bar{x}) \tag{4.35}$$

for all $x \in X \cap B$, $x \neq \bar{x}$.

As X is Φ-convex and μ is (Φ, T)-concave on X, it follows that there exists $\varphi \in \Phi$ such that $\varphi(\bar{x}, y, \lambda) \in X$ and

$$\mu(\varphi(\bar{x}, y, \lambda)) \geq T(\mu(\bar{x}), \mu(y)) \tag{4.36}$$

for each $\lambda \in [0,1]$. Moreover, from (4.34) we obtain $\varphi(\bar{x}, y, \lambda_0) \in X \cap B$ for some sufficiently small $\lambda_0 \in (0,1)$. Let $z = \varphi(\bar{x}, y, \lambda_0)$. Applying (4.36), we conclude that

$$\mu(z) \geq T(\mu(\bar{x}), \mu(y)) = T(\mu(\bar{x}), 1) = \mu(\bar{x}),$$

a contradiction to (4.35). Consequently, $\bar{x}$ must be a strict global maximizer on X. ∎

If we drop the assumption of strictness of the local maximizer in Theorem 4.49, then the statement is no longer valid. The semistrict concavity will, however, secure the result.

THEOREM 4.50 *Let X be a nonempty Φ-convex subset of $\mathbf{R}^n$, where Φ is a set of mappings $\varphi : X \times X \times [0,1] \to \mathbf{R}^n$ such that for all $x, y \in X$,*

$$\lim_{\lambda \to 0_+} \varphi(x, y, \lambda) = x.$$

Moreover, let $T : [0,1]^2 \to [0,1]$ be a t-norm, $\mu : \mathbf{R}^n \to [0,1]$ be (Φ, T)-semistrict concave on X, and $\mathrm{Core}(\mu) \cap X$ *be nonempty. If $\bar{x} \in X$ is a local maximizer of μ on X, then it is a global maximizer μ on X.*

PROOF. Suppose that $\bar{x} \in X$ is not a global maximizer. Then there exists $y \in X$, such that

$$\mu(\bar{x}) < \mu(y). \tag{4.37}$$

Moreover, we have $y \in \mathrm{Core}(\mu)$. Since $\bar{x} \in X$ is a local maximizer, there exists an open ball B with the center at $\bar{x} \in X$ such that

$$\mu(x) \le \mu(\bar{x}) \tag{4.38}$$

for all $x \in X \cap B$.

As X is Φ-convex and μ is (Φ, T)-semistrictly concave on X, it follows that there exists $\varphi \in \Phi$ such that $\varphi(\bar{x}, y, \lambda) \in X$ and by (4.37)

$$\mu(\varphi(\bar{x}, y, \lambda)) > T(\mu(\bar{x}), \mu(y)) \tag{4.39}$$

for each $\lambda \in [0,1]$. Moreover, from (4.34) we obtain $\varphi(\bar{x}, y, \lambda_0) \in X \cap B$ for some sufficiently small $\lambda_0 \in (0,1)$. Let $z = \varphi(\bar{x}, y, \lambda_0)$. Applying (4.39), we conclude that

$$\mu(z) > T(\mu(\bar{x}), \mu(y)) = T(\mu(\bar{x}), 1) = \mu(\bar{x}),$$

a contradiction to (4.38). Consequently, $\bar{x}$ must be a global maximizer on X. ■

Clearly, analogous theorems to Theorem 4.49 and 4.50 hold for local and global minimizers of (Φ, S)-convex functions.

Chapter 5

AGGREGATION OPERATORS

1. Introduction

Aggregation refers to the process of combining values into a single value so that the final result of aggregation takes into account in a given form all the individual aggregated values. This chapter serves as a theoretical background for applications mainly in the area of decision analysis, decision making or decision support. In decision making, values to be aggregated are typically preference or satisfaction degrees. A preference degree, e.g., $v(A, B)$ tells to what extent an alternative A is preferred to an alternative B. This way, however, will not be followed here. In this chapter the values are understood and interpreted as satisfaction degrees which express to what extent a given alternative is satisfactory with respect to a given criterion - a given real-valued function, or as a kind of distance to a prototype which may represent the ideal alternative for the decision maker. Depending on concrete applications, values to be aggregated can be also interpreted as confidence degrees in the fact that a given alternative is true, or as expert's opinions, similarity degrees, etc., see, e.g., [29].

Once values on a scale, e.g., the unit interval $[0, 1]$, are given we can aggregate them and obtain a new value defined on the same scale, but this can be done in many different ways according to what is expected from the aggregation mapping called aggregation operator. Aggregation operators can be roughly divided into three classes, each possessing very distinct behavior and semantics.

Operators of the first class, *conjunctive type operators*, combine values as if they were related by a logical "and" operation. In other words, the result of combination is high, if all the values are high. Triangular norms are examples for doing conjunctive type aggregations.

On the other hand, *disjunctive type operators* combine values as an "or" operation, so that the result of aggregation is high if some values are high. Similarly to the first class, the most common examples of disjunctive type operators are triangular conorms.

Between conjunctive and disjunctive type operators, there is a room for a third class of aggregation operators which could be called *averaging type operators*. They are usually located between minimum and maximum, which are the bounds of the t-norms and t-conorms. Averaging type operators have the property that low values of some criteria can be compensated by high values of the other criteria functions.

There are of course other operators which do not fit into any of these classes we shall also deal with them later on.

2. Definition and Basic Properties

When aggregating data in applications, we assign to each tuple of elements a unique real number. For this purpose, both t-norms and t-conorms are rather special operators on the unit interval $[0, 1]$. However, there exist other useful operations related to and generalizing t-norms or t-conorms, either on the unit interval or on arbitrary closed subinterval $[a, b]$ of the extended real line $\mathbf{R}$. Because of the natural correspondence between $[a, b]$ and $[0, 1]$, each result for operations on the interval $[a, b]$ can be transformed into a result for operators on $[0, 1]$ and vice versa. Therefore, the discussion about aggregation operators on $[0, 1]$ is sufficiently general, at least from theoretical point of view. In many cases, general aggregation operators can be derived from n-ary operations on $[0, 1]$, for instance from t-norms.

DEFINITION 5.1 *An* aggregation operator A *is a sequence* $\{A_n\}_{n=1}^{\infty}$ *of mappings (called* aggregating mappings*)*

$$A_n : [0, 1]^n \to [0, 1],$$

satisfying the following properties:

(i) $A_1(x) = x$ *for each* $x \in [0, 1]$*;*

(ii) $A_n(x_1, x_2, \ldots, x_n) \leq A_n(y_1, y_2, \ldots, y_n)$*, whenever* $x_i \leq y_i$ *for each* $i = 1, 2, \ldots, n$*, and every* $n = 2, 3, \ldots$*;*

(iii) $A_n(0, 0, \ldots, 0) = 0$ *and* $A_n(1, 1, \ldots, 1) = 1$ *for every* $n = 2, 3, \ldots$.

Condition (i) means that A_1 is an identity unary operation, (ii) says that aggregating mapping A_n is nondecreasing in all of its arguments x_i, and condition (iii) represents natural boundary requirements. Some other mathematical properties can be requested for an aggregation operators, we list some of them in the following definition.

DEFINITION 5.2 *Let $A = \{A_n\}_{n=1}^{\infty}$ be an aggregation operator.*

(i) *The aggregation operator A is called* commutative, idempotent, nilpotent, strictly monotone *or* continuous, *if, for each $n \geq 2$, the aggregating mapping A_n is commutative, idempotent, strictly monotone or continuous, respectively. The aggregation operator A is called* strict, *if A_n is strictly monotone and continuous for all $n \geq 2$.*

(ii) *The aggregation operator A is called* associative, *if, for all $m, n \geq 2$ and all tuples $(x_1, x_2, \ldots, x_m) \in [0,1]^m$ and $(y_1, y_2, \ldots, y_n) \in [0,1]^n$, we have*

$$A_{m+n}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = A_2(A_m(x_1, x_2, \ldots, x_m), A_n(y_1, y_2, \ldots, y_n)).$$

(iii) *The aggregation operator A is called* decomposable, *if, for all $m, n \geq 2$ and all tuples $(x_1, \ldots, x_m) \in [0,1]^m$ and $(y_1, \ldots, y_n) \in [0,1]^n$, we have*

$$A_{m+n}(x_1, \ldots, x_m, y_1, \ldots, y_n) = A_{m+n}(A_m(x_1, \ldots, x_m), \ldots, A_m(x_1, \ldots, x_m), y_1, \ldots, y_n) \quad (5.1)$$

where, in the right side, the term $A_m(x_1, x_2, \ldots, x_m)$ occurs m times.

(iv) *The aggregation operator A is called* compensative, *if, for $n \geq 2$ and for all tuples $(x_1, x_2, \ldots, x_n) \in [0,1]^n$, the following inequalities hold:*

$$T_M(x_1, x_2, \ldots, x_n) \leq A_n(x_1, x_2, \ldots, x_n) \leq S_M(x_1, x_2, \ldots, x_n). \quad (5.2)$$

The commutativity and associativity properties allow extending of t-norms and t-conorms to n-ary operations, with $n > 2$. For instance, for a t-norm T, its extension T^n to n arguments has been defined by formula (4.9). Therefore, a sequence $\{T^n\}_{n=1}^{\infty}$, where T^1 is the identity mapping, defines an aggregation operator, and T^n are its aggregating mappings. For the sake of simplicity, when there is no danger of a confusion, we call this aggregation operator also a t-norm and denote it by the original symbol T. In other words, when speaking about a t-norm T, or t-conorm S, as an aggregating operator, we always have in mind the corresponding sequence $\{T^n\}_{n=1}^{\infty}$, or $\{S^n\}_{n=1}^{\infty}$, respectively. Recall also, that for the same reason, we shall sometimes leave the index n in the aggregating mappings A_n. Considering this convention in the following propositions, we obtain some characterizations of the above defined properties.

PROPOSITION 5.3 *Each t-norm and each t-conorm is a commutative and associative aggregation operator. The minimum T_M is the only idempotent t-norm, but it is not strict. The product norm T_P is strict, but not nilpotent.*

Łukasiewicz t-norm T_L is both strict and nilpotent. The drastic product T_D is nilpotent, but not continuous.

Analogical properties hold also for t-conorms S_M, S_P, S_L and S_D. A transformation of an aggregation operator by means of a monotone bijection from $[0,1]$ to $[0,1]$ yields again an aggregation operator. We have the following proposition the proof of which is elementary.

PROPOSITION 5.4 *Let $A = \{A_n\}_{n=1}^{\infty}$ be an aggregation operator and let $\psi : [0,1] \to [0,1]$ be a strictly increasing or strictly decreasing bijection. Then $A^\psi = \{A_n^\psi\}_{n=1}^{\infty}$ defined by*

$$A_n^\psi(x_1, x_2, \dots, x_n) = \psi^{-1}(A_n(\psi(x_1), \dots, \psi(x_n)))$$

for all $n = 1, 2, \dots$, and all tuples $(x_1, x_2, \dots, x_n) \in [0,1]^n$, is an aggregation operator.

3. Continuity Properties

Continuity of aggregation operators play an important role in applications, as we shall see in Chapter 6. As far as t-norms and t-conorms are concerned, we already know that T_M, T_P, T_L are continuous t-norms, whereas the drastic product T_D is not. The same holds for the dual t-conorm S_M, S_P, S_L and S_D. In the following proposition we show that for continuity of commutative aggregation operators it is sufficient that they are continuous in only one variable.

PROPOSITION 5.5 *Let $A = \{A_n\}_{n=1}^{\infty}$ be a commutative aggregation operator. The operator A is continuous if and only if, for each $n = 1, 2, \dots$, mapping A_n is continuous in its first variable x_1, i.e., if for each n and $x_2, \dots, x_n \in [0,1]$ the one variable function $A(\cdot, x_2, \dots, x_n)$ is continuous on $[0,1]$.*

PROOF. 1. If a function A_n is continuous on $[0,1]^n$, then it is continuous in each variable.

2. Conversely, by commutativity, A_n is continuous individually in all variables. We prove the statement by induction according to n.

Evidently, the statement is true for $n = 1$. Suppose that the statement is true for some $n > 1$, we will show that it is true for $n + 1$. Consider an aggregating mapping $A_{n+1}(x_1, x_2, \dots, x_n, x_{n+1})$ which is continuous in its first variable x_1, fix an arbitrary vector $\mathbf{x}'^0 = (\mathbf{x}^0, x_{n+1}^0) = (x_1^0, x_2^0, \dots, x_n^0, x_{n+1}^0) \in [0,1]^{n+1}$ and define

$$A_n'(x_1, x_2, \dots, x_n) = A_{n+1}(x_1, x_2, \dots, x_n, x_{n+1}^0).$$

Clearly A_n' is a commutative aggregating mapping and by induction assumption it is continuous. Let $\{\mathbf{x}_k'\}_{k=1}^{\infty}$ be a sequence converging to $\mathbf{x}'^0$, i.e.

$\mathbf{x}'_k \in [0,1]^{n+1}$, $\mathbf{x}'_k \to \mathbf{x}'^0$. We can find two sequences $\{\mathbf{a}'_k\}_{k=1}^{\infty}$, $\{\mathbf{b}'_k\}_{k=1}^{\infty}$, $\mathbf{a}'_k, \mathbf{b}'_k \in [0,1]^{n+1}$, the first sequence is non-decreasing the second one is non-increasing in all $n+1$ components, such that

$$\mathbf{a}'_k \leq \mathbf{x}'_k \leq \mathbf{b}'_k, \quad \text{for all } k = 1, 2, \ldots, \tag{5.3}$$

and

$$\mathbf{a}'_l \to \mathbf{x}'^0, \mathbf{b}'_l \to \mathbf{x}'^0 \quad \text{for } l \to +\infty. \tag{5.4}$$

Formulae (5.3) and (5.4) can be unfolded into two parts

$$\begin{aligned} \mathbf{a}_k \leq \mathbf{x}_k \leq \mathbf{b}_k, & \qquad \text{for all } k = 1, 2, \ldots, \\ a_{n+1,k} \leq x_{n+1,k} \leq b_{n+1,k}, & \qquad \text{for all } k = 1, 2, \ldots, \end{aligned}$$

and

$$\begin{aligned} \mathbf{a}_l \to \mathbf{x}^0, \ \ \mathbf{b}_l \to \mathbf{x}^0 & \qquad \text{for } l \to +\infty, \\ a_{n+1,l} \to x^0_{n+1}, \ \ b_{n+1,l} \to x^0_{n+1} & \qquad \text{for } l \to +\infty. \end{aligned}$$

Let $\varepsilon > 0$. Then by continuity of A_{n+1} in variable x_{n+1} at x^0_{n+1}, there exists l_0 such that for $l > l_0$:

$$\begin{aligned} A_{n+1}(\mathbf{x}^0, x^0_{n+1}) - \varepsilon < A_{n+1}(\mathbf{x}^0, a_{n+1,l_0}) &\leq A_{n+1}(\mathbf{x}^0, x_{n+1,l}) \\ &\leq A_{n+1}(\mathbf{x}^0, b_{n+1,l_0}) < A_{n+1}(\mathbf{x}^0, x^0_{n+1}) + \varepsilon. \end{aligned}$$

By induction assumption, functions $A_{n+1}(\cdot, a_{n+1,l_0})$ and $A_{n+1}(\cdot, b_{n+1,l_0})$ are continuous, there is a number m_0 such that for all $m > m_0$, $l > l_0$ we get by monotonicity of A_{n+1}

$$\begin{aligned} A_{n+1}(\mathbf{x}^0, a_{n+1,l_0}) - \varepsilon < A_{n+1}(\mathbf{a}_{m_0}, a_{n+1,l_0}) &\leq A_{n+1}(\mathbf{x}_m, x_{n+1,l}) \\ &\leq A_{n+1}(\mathbf{b}_{m_0}, b_{n+1,l_0}) < A_{n+1}(\mathbf{x}^0, b_{n+1,l_0}) + \varepsilon. \end{aligned}$$

Putting $k_0 = \max\{m_0, l_0\}$, then for all $k > k_0$ we obtain

$$A_{n+1}(\mathbf{x}^0, x^0_{n+1}) - 2\varepsilon < A_{n+1}(\mathbf{x}_k, x_{n+1,k}) < A_{n+1}(\mathbf{x}^0, x^0_{n+1}) + 2\varepsilon,$$

proving that $A_{n+1}(\mathbf{x}'_k) \to A_{n+1}(\mathbf{x}'^0)$ for $k \to +\infty$, that is, the aggregation operator A is continuous in $\mathbf{x}'^0 \in [0,1]^{n+1}$. ■

Similarly to the situation in the case of continuity, both lower and upper semicontinuity of aggregation operators can be described by semicontinuity in only one variable.

PROPOSITION 5.6 *Let $A = \{A_n\}_{n=1}^{\infty}$ be a commutative aggregation operator. The operator A is upper semicontinuous - USC (lower semicontinuous - LSC) if and only if, for each $n = 1, 2, \ldots$, A_n is USC (LSC) in its first variable,*

i.e., if for each $x_2, \ldots, x_n \in [0,1]$ *the one variable function* $A_n(\cdot, x_2, \ldots, x_n)$ *is USC (LSC) on* $[0,1]$, *respectively.*

PROOF. 1. If a function A_n is USC (LSC) on $[0,1]^n$, then it is USC (LSC) in each variable.
2. Conversely, by commutativity, A_n is USC (LSC) individually in all variables. We prove the statement by induction according to n.

Evidently, the statement is true for $n = 1$. Suppose that the statement is true for some $n > 1$. We show that it is true for $n + 1$. The rest of the proof is analogous to the proof of Proposition 5.5. To prove USC, we use the sequence $\{\mathrm{b}'_k\}_{k=1}^{\infty}$ from the proof of Proposition 5.5; on the other hand, to prove LSC of A, we utilize $\{\mathrm{a}'_k\}_{k=1}^{\infty}$, and we apply the same inequalities as in the proof of Proposition 5.5. ∎

By monotonicity of an aggregation operator A, the left (right) continuity of A is equivalent to the LSC (USC) of A. Note also that the left and right continuity mean exactly the interchangeability of the supremum and infimum, respectively, with the application of the aggregation operator.

4. Averaging Aggregation Operators

Between conjunctive and disjunctive type operators, t-norms and t-conorms, which have been investigated thoroughly in Chapter 4, there is a room for another class of aggregation operators of averaging type. They are located between minimum and maximum satisfying inequalities (5.2). Averaging type operators have the property that low values of some criteria can be compensated by high values of the other criteria.

4.1. Compensative Aggregation Operators

May be even more popular aggregation operators than t-norms and t-conorms are the means: the *arithmetic mean* $M = \{M_n\}_{n=1}^{\infty}$, the *geometric mean* $G = \{G_n\}_{n=1}^{\infty}$, the *harmonic mean* $H = \{H_n\}_{n=1}^{\infty}$ and the *root-power mean* $M^{(\alpha)} = \{M_n^{(\alpha)}\}_{n=1}^{\infty}$, given by, respectively,

$$M_n(x_1, x_2, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i, \tag{5.5}$$

$$G_n(x_1, x_2, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i\right)^{1/n} \tag{5.6}$$

$$H_n(x_1, x_2, \ldots, x_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}, \tag{5.7}$$

$$M_n^{(\alpha)}(x_1, x_2, \ldots, x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha}\right)^{1/\alpha}, \quad \alpha \neq 0. \tag{5.8}$$

All these operators are commutative, idempotent and continuous, none of them is associative. The root-power mean operators $M^{(\alpha)}$, $\alpha \geq 0$, are strict, whereas G and H are not strict. Notice that $M = M^{(1)}$ and $H = M^{(-1)}$. It can be verified that

$$\begin{aligned}
M_n^{(0)}(x_1, x_2, \ldots, x_n) &= \lim_{\alpha \to 0} M^{(\alpha)}(x_1, x_2, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \\
M_n^{(-\infty)}(x_1, x_2, \ldots, x_n) &= \lim_{\alpha \to -\infty} M^{(\alpha)}(x_1, x_2, \ldots, x_n) \\
&= \min\{x_i \mid i = 1, \ldots, n\}, \\
M_n^{(+\infty)}(x_1, x_2, \ldots, x_n) &= \lim_{\alpha \to +\infty} M^{(\alpha)}(x_1, x_2, \ldots, x_n) \\
&= \max\{x_i \mid i = 1, \ldots, n\}.
\end{aligned}$$

The next proposition says that the operators (5.5) - (5.8) are all compensative. It says even more, namely, that the class of idempotent aggregation operators is exactly the same as the class of compensative ones. The proof of this result is elementary and can be found in [32].

PROPOSITION 5.7 *Let $A = \{A_n\}_{n=1}^{\infty}$ be an aggregation operator. Then, A is idempotent if and only if A is compensative.*

In the following propositions we clarify the relationships between some other properties introduced in Definition 5.2, particularly, between the decomposability and associativity. The proofs can be found also in [32].

PROPOSITION 5.8 *Let $A = \{A_n\}_{n=1}^{\infty}$ be a continuous and commutative aggregation operator. Then A is compensative, strict and decomposable, if and only if for all $x_1, x_2, \ldots, x_n \in [0, 1]$*

$$A_n(x_1, x_2, \ldots, x_n) = \psi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} \psi(x_i)\right), \tag{5.9}$$

with a continuous strictly monotone function $\psi : [0, 1] \to [0, 1]$.

The aggregation operator (5.9) is called the *generalized mean.* It covers a wide range of popular means including those of (5.5) - (5.8). The minimum T_M and the maximum S_M are the only associative and decomposable compensative aggregation operators.

4.2. Order-Statistic Aggregation Operators

Let $x = (x_1, x_2, \dots, x_n)$ be a point in $\mathbf{R}^n$. The point in $\mathbf{R}^n$ which we obtain by arranging the coordinates of x in non-decreasing order is denoted by $x^{\uparrow} = (x_{(1)}, x_{(2)}, \dots, x_{(n)})$. Notice that there may exist several permutations π of the index set $\{1, 2, \dots, n\}$ with the property $x_{\pi(i)} = x_{(i)}$. For example if $x = (4, 1, 4)$, then $x^{\uparrow} = (1, 4, 4)$, the following two permutations have the property: $\pi(1) = 2, \pi(2) = 1, \pi(3) = 3$, and $\pi'(1) = 2, \pi'(2) = 3, \pi'(3) = 1$.

DEFINITION 5.9 *Let $k \in \{1, 2, \dots\}$. The k-order statistic aggregation operator $\mathrm{OS}^k = \{\mathrm{OS}_n^k\}_{n=1}^{\infty}$ is defined as follows*

$$\mathrm{OS}_n^k(x_1, x_2, \dots, x_n) = \begin{cases} x_{(n)} & \text{if } k > n, \\ x_{(k)} & \text{if } k \leq n, \end{cases}$$

for all $x_1, x_2, \dots, x_n \in [0, 1]$.

Particular cases are the minimum ($k = 1$), the maximum ($k = n$), and the *median aggregation operator* $\mathrm{OS}^{\mathrm{med}}$, defined by

$$\mathrm{OS}_n^{\mathrm{med}}(x_1, x_2, \dots, x_n) = \begin{cases} x_{\left(\frac{n+1}{2}\right)} & \text{if } n \text{ is odd}, \\ \frac{1}{2}\left(x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n+2}{2}\right)}\right) & \text{if } n \text{ is even}. \end{cases} \tag{5.10}$$

The proofs of the two following propositions can be found in [114].

PROPOSITION 5.10 *Let A be a k-order statistic aggregation operator. Then A is compensative, commutative and continuous.*

PROPOSITION 5.11 *Let $A = \{A_n\}_{n=1}^{\infty}$ be an aggregation operator. Then A is compensative, commutative, continuous and associative if and only if for each $\alpha \in (0, 1)$ and each $x_1, x_2, \dots, x_n \in [0, 1]$*

$$A_n(x_1, x_2, \dots, x_n) = \mathrm{OS}_3^{\mathrm{med}}(x_{(1)}, x_{(n)}, \alpha).$$

4.3. Order Weighted Averaging Operators

So called the order weighted averaging (OWA) operators belong to the class of average-type compensative aggregation operators. These operators allow for an adjustment of the degree of "and-ing" and "or-ing", which means that the character of the operators can be adjusted either more to the conjunction-type or more to the disjunction-type aggregation operators.

DEFINITION 5.12 *Let $W = (w_1, w_2, \dots, w_n)$ be a* weighting vector *with $w_i \geq 0$, $i = 1, 2, \dots, n$, and $\sum_{i=1}^{n} w_i = 1$. A mapping $\mathrm{OWA}_W^n : [0, 1]^n \to [0, 1]$ defined for each $x_1, x_2, \dots, x_n \in [0, 1]$ by*

$$\mathrm{OWA}_W^n(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} w_i x_{(n-i+1)}$$

is called the OWA operator of dimension n with associated vector W.

Notice, that OWA operators by Definition 5.12 are not aggregating operators according to Definition 5.1. In Definition 5.12 the operator is strongly associated with the n-dimensional weighting vector W for a given n, and cannot be extended to vectors of other dimensions. A fundamental aspect of the operation (5.10) is the re-ordering step, in particular, within a sum a value x_i is not associated with the corresponding weight w_i, but rather a weight is associated with a particular ordered position of the value. This ordering step introduces a nonlinearity into the aggregation operation.

EXAMPLE 5.13 Let

$$W = (w_1, w_2, \ldots, w_5) = (0.2, 0.1, 0.1, 0.4, 0.2),$$
$$x = (x_1, x_2, \ldots, x_5) = (0.9, 0.3, 0.6, 0.3, 0.2).$$

Then $x^{\uparrow} = (x_{(1)}, x_{(2)}, \ldots, x_{(5)}) = (0.2, 0.3, 0.3, 0.6, 0.9)$ and

$$\begin{aligned}
\mathrm{OWA}_W^5(x_1, x_2, \ldots, x_5) &= \mathrm{OWA}_W^5(0.9, 0.3, 0.6, 0.3, 0.2) \\
&= w_1 x_{(5)} + w_2 x_{(4)} + \ldots + w_5 x_{(1)} \\
&= 0.2 \cdot 0.9 + 0.1 \cdot 0.6 + 0.1 \cdot 0.3 \\
&\quad +0.4 \cdot 0.3 + 0.2 \cdot 0.2 \\
&= 0.43.
\end{aligned}$$

□

Some important special cases of OWA operators are defined as follows:

- For $W^* = (w_1, w_2, \ldots, w_n)$ such that $w_1 = 1$ and $w_i = 0$, for $i = 2, 3, \ldots, n$, the OWA operator equals to the nth aggregating mapping S_M^n of the maximum aggregation operator $S_M = \{S_M^n\}_{n=1}^{\infty}$, i.e., $\mathrm{OWA}_{W^*}^n = S_M^n$.
- For $W_* = (w_1, w_2, \ldots, w_n)$ such that $w_n = 1$ and $w_i = 0$, for $i = 1, 2, \ldots, n-1$, the OWA operator equals to the nth aggregating mapping T_M^n of the minimum aggregation operator $T_M = \{T_M^n\}_{n=1}^{\infty}$, i.e., $\mathrm{OWA}_{W^*}^n = T_M^n$.
- For $W_M = (w_1, w_2, \ldots, w_n)$ such that $w_i = 1/n$, for all $i = 1, 2, \ldots, n$, the OWA operator equals to the nth aggregating mapping M_n of the arithmetic mean aggregation operator $M = \{M_n\}_{n=1}^{\infty}$, i.e., $\mathrm{OWA}_{W_M}^n = M_n$.
- For $W_k = (w_1, w_2, \ldots, w_n)$ such that $w_{n-k+1} = 1$ and $w_i = 0$, for all other weights, the OWA operator equals to the nth aggregating mapping OS_n^k of the kth order statistic aggregation operator $\mathrm{OS}^k = \{\mathrm{OS}_n^k\}_{n=1}^{\infty}$, i.e., $\mathrm{OWA}_{W_k}^n = \mathrm{OS}_n^k$.

- For $W_{\mathrm{med}} = (w_1, w_2, \ldots, w_n)$ such that $w_{\frac{n+1}{2}} = 1$ and $w_i = 0$, for all other weights, if n is odd; $w_{\frac{n}{2}} = w_{\frac{n}{2}+1} = 0.5$ and $w_i = 0$, for all other weights, if n is even. The OWA operator equals to the nth aggregating mapping $\mathrm{OS}_n^{\mathrm{med}}$ of the median aggregation operator $\mathrm{OS}^{\mathrm{med}} = \{\mathrm{OS}_n^{\mathrm{med}}\}_{n=1}^{\infty}$, i.e., $\mathrm{OWA}_{W_k}^n = \mathrm{OS}_n^{\mathrm{med}}$.

The next proposition will characterize OWA operators, the proof can be found in [32].

PROPOSITION 5.14 *Let A be an* OWA *operator. Then A is compensative, commutative and continuous.*

5. Sugeno and Choquet Integrals

Generally speaking, the integral of a function over a set represents, in a sense, are average value of that function. In case of a finite set with suitable normalization, an integral may represent an aggregation operator. Extending the well known concept of Lebesque integral, that is, an integral with respect to a measure, G. Choquet in [79] and M. Sugeno in [59] proposed integrals which have found applications in various areas: fuzzy sets, utility theory, decision analysis, game theory, engineering, etc. Here we restrict ourselves to the finite case avoiding unnecessary mathematical developments with the main aim to present the integrals as general aggregation tools covering the concepts of the preceding sections. The reader asking more details about the integrals is referred to [37] and [36]. In order to define new integrals we need to generalize the traditional concept of a measure. The following concept belongs to M. Sugeno; see [120].

DEFINITION 5.15 *Let Z be a nonempty finite set, and let 2^Z denote the set of all subsets of Z. A* fuzzy measure *on Z is a set function $\varphi : 2^Z \to [0,1]$ satisfying the following properties:*

(i) $\varphi(\emptyset) = 0$, $\varphi(X) = 1$.

(ii) *If $A, B \in 2^Z$ and $A \subset B$, then $\varphi(A) \leq \varphi(B)$.*

In a decision analysis setting, the set $\mathcal{N} = \{1, 2, \ldots, n\}$ can be interpreted as a set of decision criteria. By X we denote a real-valued function on $\mathcal{N}$. For the value of the ith criterion, we use the notation $X(i) = x_i$, $i \in \mathcal{N}$. Then for $A \in 2^{\mathcal{N}}$, the value $\varphi(A)$ of a fuzzy measure φ can be viewed as the weight of the subset A of the criteria set $\mathcal{N}$. Thus, in addition to the usual weights on criteria taken separately, weights on any combination of criteria are also defined by the fuzzy measure φ.

A fuzzy measure is said to be *additive* if

$$\varphi(A \cup B) = \varphi(A) + \varphi(B) \text{ whenever } A, B \in 2^{\mathcal{N}}, A \cap B = \emptyset,$$

it is said to be *superadditive* (resp. *subadditive*) if

$$\varphi(A \cup B) \geq \varphi(A) + \varphi(B) \text{ whenever } A, B \in 2^{\mathcal{N}}, A \cap B = \emptyset,$$

resp.

$$\varphi(A \cup B) \leq \varphi(A) + \varphi(B) \text{ whenever } A, B \in 2^{\mathcal{N}}, A \cap B = \emptyset.$$

If a fuzzy measure φ is additive, then it suffices to define the n individual weights $\varphi(\{1\}), \varphi(\{2\}), \ldots, \varphi(\{n\})$ to define entirely the measure φ. However, in a general case one has to define $2^n - 2$ weights corresponding to the $2^n - 2$ subsets of $\mathcal{N}$, except $\emptyset$ and $\mathcal{N}$.

DEFINITION 5.16 *Let φ be a fuzzy measure on $\mathcal{N} = \{1, 2, \ldots, n\}$ and X be a real-valued function on $\mathcal{N}$. The* Sugeno integral S^n_φ of X with respect to φ *is defined by*

$$S^n_\varphi(x_1, x_2, \ldots, x_n) = \max\left\{\min\{x_{(i)}, \varphi(A_{(i)})\} \mid i \in \mathcal{N}\right\}, \qquad (5.11)$$

where, for every $i \in \mathcal{N}$, $x_i = X(i)$ and $A_{(i)} = \{i, i+1, \ldots, n\}$. The Sugeno integral is also denoted by $(S)\ \int_{\mathcal{N}} X d\varphi$.

DEFINITION 5.17 *Let φ be a fuzzy measure on $\mathcal{N} = \{1, 2, \ldots, n\}$, and X be a real-valued function on $\mathcal{N}$. The* Choquet integral C^n_φ of X with respect to φ *is defined by*

$$C^n_\varphi(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} (x_{(i)} - x_{(i-1)})\varphi\left(A_{(i)}\right), \qquad (5.12)$$

where, for every $i \in \mathcal{N}$, $x_i = X(i)$ and $A_{(i)} = \{i, i+1, \ldots, n\}$. The Choquet integral is also denoted by $(C)\ \int_{\mathcal{N}} X d\varphi$.

Sugeno and Choquet integrals are essentially different in nature, since the former is based on non-linear operators maximization and minimization, and the latter on usual linear operators. More general definitions of these types integrals exist but will not be considered here, see, e.g., [37] or [36]. In Definitions 5.16 and 5.17, we follow a traditional concept of integral, however, in view of Definition 5.1, Sugeno and Choquet integrals can be considered as the aggregation operators $S_\varphi = \{S^n_\varphi\}_{n=1}^{\infty}$ and $C_\varphi = \{C^n_\varphi\}_{n=1}^{\infty}$, respectively, provided the range of X is in $[0, 1]$. Some properties of the integrals are presented in the following proposition; see [37] or [32].

PROPOSITION 5.18 *Let φ be a fuzzy measure on $\mathcal{N} = \{1, 2, \ldots, n\}$, and X be a function mapping $\mathcal{N}$ into $[0, 1]$. The Sugeno and Choquet integrals defined by (5.11) and (5.12), respectively, are compensative and continuous aggregation operators.*

The connection between OWA operators and various operators (in particular k-order statistic ones) has been already mentioned in the preceding section. We turn now to Sugeno and Choquet integrals. They include as particular cases many of the operators defined previously. Main results are summarized below in the four propositions; their proofs can be found in [37] or [32].

PROPOSITION 5.19 *Let φ be an additive fuzzy measure on $\mathcal{N} = \{1, 2, ..., n\}$ and X be a function mapping $\mathcal{N}$ into $[0, 1]$. Then the Choquet integral C_φ^n of X with respect to φ coincides with the weighted arithmetic mean aggregating mapping $M_n^w(x_1, x_2, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^n w_i x_i$, whose weights are $w_i = \varphi(\{i\}), i \in \mathcal{N}$.*

PROPOSITION 5.20 *Let $\mathcal{N} = \{1, 2, \ldots, n\}$, X be a function mapping $\mathcal{N}$ into $[0, 1]$, and let φ be the fuzzy measure on $\mathcal{N}$ defined by*

$$\varphi(A) = \begin{cases} 0 & \textit{if}\, \mathrm{Card}(A) \le n - k, \\ 1 & \textit{otherwise}, \end{cases}$$

where $k \in \{1, 2, \ldots, n-1\}$ and $\mathrm{Card}(A)$ denotes the cardinality of A. Then the Choquet integral C_φ^n of X with respect to φ is a k-order statistic aggregating mapping OS_n^k.

PROPOSITION 5.21 *Let OWA_W^n be an OWA operator with the weighting vector $W = (w_1, w_2, ..., w_n)$, and X be a function mapping $\mathcal{N} = \{1, 2, ..., n\}$ into $[0, 1]$. Then OWA_W^n is equal to the Choquet integral C_φ^n of X with respect to fuzzy measure φ defined by*

$$\varphi(A) = \sum_{j=1}^{i} w_j \quad \textit{for each } A \in 2^{\mathcal{N}} \textit{ such that } \mathrm{Card}(A) = i.$$

PROPOSITION 5.22 *Let C_φ^n be a commutative Choquet integral of $X : \mathcal{N} \to [0, 1]$ with respect to a fuzzy measure φ. Then C_φ^n is an OWA operator with the weighting vector $W = (w_1, w_2, \ldots, w_n)$ defined by $w_i = \varphi(A_i) - \varphi(A_{i-1})$, for $i = 2, 3, \ldots, n$ and $w_1 = 1 - \sum_{i=2}^n w_i$, where A_i denotes any subset of $\mathcal{N}$ such that $\mathrm{Card}(A_i) = i$.*

The main relationships between the investigated aggregation operators are depicted in Figure 5.1. In this figure, the OWA operators are depicted by a dotted line denoting the fact that OWA operators are not aggregation operators. However, if we consider only the nth aggregating mapping of the operators, then the figure remains valid and OWA operators could be included.

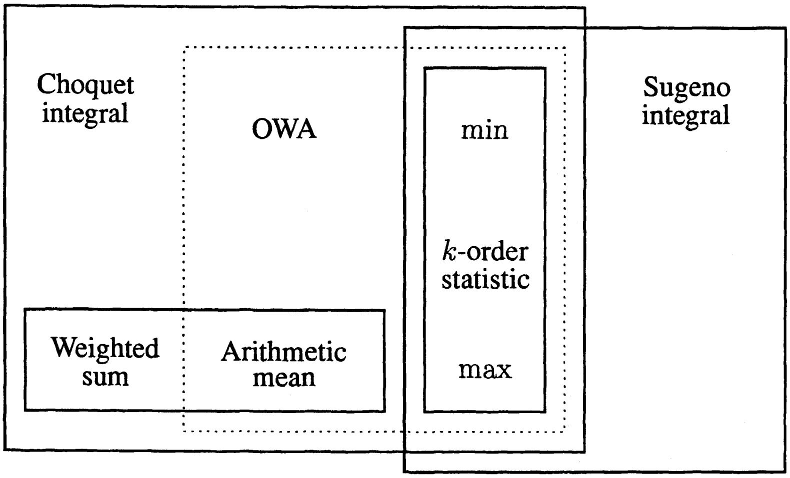

Figure 5.1.

6. Other Aggregation Operators

Conjunctive, disjunctive and compensative operators form large disjoint classes of operators on $[0,1]$, but there are some operators which belong to none of these classes. These operators are often more or less of the average type, but they may be extended beyond the minimum and maximum operators. Note that the operators defined below, i.e., symmetric sums and compensatory operators, are aggregation operators in the sense of Definition 5.1.

DEFINITION 5.23 *The* symmetric sum *is a continuous function* $s : [0,1]^2 \to [0,1]$ *satisfying the following conditions:*

$$\begin{aligned} s(a,b) &= s(b,a) \quad \textit{for all } a,b \in [0,1], && (5.13)\\ s(a,b) &\leq s(c,d) \quad \textit{if } a \leq c \textit{ and } b \leq d,\ a,b,c,d \in [0,1], && (5.14)\\ 1 - s(a,b) &= s(1-a, 1-b) \quad \textit{for all } a,b \in [0,1], && (5.15)\\ s(0,0) &= 0, s(1,1) = 1. && (5.16) \end{aligned}$$

Conditions (5.13) and (5.14) are commutativity and monotonicity, respectively; condition (5.15) is called *self-duality* and condition (5.16) is the usual boundary condition. Self-duality condition (5.15) can be extended by using any strong negation instead of standard negation $N(x) = 1 - x$. Symmetric sums can be expressed in a special form as follows; see [32].

PROPOSITION 5.24 *Let* $s : [0,1]^2 \longrightarrow [0,1]$ *be a symmetric sum. Then for all* $a, b \in [0,1]$

$$s(a,b) = \frac{g(a,b)}{g(a,b) + g(1-a, 1-b)},$$

where $g : [0,1]^2 \longrightarrow [0,1]$ *is a continuous, nondecreasing function such that* $g(a,b) = 0$ *if* $(a,b) = (0,0)$, *and* $g(a,b) > 0$ *otherwise.*

By a straightforward way, similar to that applied t-norms, the symmetric sum s can be extended for n variables, $n > 2$, denoted by s_n, to serve as a continuous commutative aggregation operator $S_s = \{s_n\}_{n=1}^{\infty}$.

A large class of aggregation operators can be constructed as a combination of t-norms and t-conorms. From the application point of view, there exist suggestions for using so-called *compensatory operators* (being distinct from compensative ones). To avoid certain defects of t-norms and t-conorms with respect to compensation of small values of some criteria by large values of the other ones, two classes of compensatory operators were suggested, see [139], and generalized later in [36] for t-norms and t-conorms. The first is the class of *exponential-type operators*, defined for a t-norm T, a t-conorm S and a parameter $\gamma \in [0,1]$ as follows:

$$E_{\gamma,n}^{T,S}(x_1, x_2, \ldots, x_n) = (T(x_1, x_2, \ldots, x_n))^{1-\gamma}(S(x_1, x_2, \ldots, x_n))^{\gamma}.$$

The second class of compensatory operators is the class of *convex-linear-type operators*, defined similarly by

$$L_{\gamma,n}^{T,S}(x_1, x_2, \ldots, x_n) = (1-\gamma)T(x_1, x_2, \ldots, x_n) + \gamma S(x_1, x_2, \ldots, x_n).$$

Both the exponential-type and convex-linear-type aggregation operators $E_{\gamma}^{T,S}$ and $L_{\gamma}^{T,S}$ are continuous, commutative and idempotent if and only if the corresponding t-norm T as well as t-conorm S are continuous and idempotent, respectively. However, they are associative only if $\gamma \in \{0, 1\}$, in which case they coincide with T and S, respectively. In the literature, a nonlinear combination of T and S has been also proposed and studied; see, e.g., [57].

7. Aggregation of Functions

In this section we shall investigate the problem of aggregation of several generalized quasiconcave functions (i.e. upper-starshaped or T-quasiconcave, or quasiconcave). We will look for sufficient conditions which secure some quasiconcave properties. The simple result of this type reads as follows.

PROPOSITION 5.25 *Let X be a convex subset of $\mathbf{R}^n$, and let $\mu_i : \mathbf{R}^n \to [0,1]$, $i = 1, 2, \ldots, m$, be upper normalized and T_D-quasiconcave on X and such that* $\mathrm{Core}(\mu_1) \cap \cdots \cap \mathrm{Core}(\mu_m) \neq \emptyset$. *Let $A_m : [0,1]^m \to [0,1]$ be an aggregating mapping. Then $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by*

$$\psi(x) = A_m(\mu_1(x), \ldots, \mu_m(x))$$

is upper-starshaped on X.

PROOF. Let $\alpha \in \mathbf{R}$, and let $\bar{x} \in \mathrm{Core}(\mu_1) \cap \cdots \cap \mathrm{Core}(\mu_m)$. We prove that $U(\psi, \alpha)$ is starshaped by demonstrating that $\bar{x} \in \mathrm{Ker}(U(\psi, \alpha))$. To show this, it is sufficient to demonstrate that, for an arbitrary point x from $U(\psi, \alpha)$, the whole segment connecting x and $\bar{x}$ belongs to $U(\psi, \alpha)$. Take arbitrarily $x \in U(\psi, \alpha)$, $\lambda \in (0, 1)$ and set $z = \lambda x + (1 - \lambda)\bar{x}$. As μ_i, $i = 1, 2, \ldots, m$, are T_D-quasiconcave on X and normalized, it follows by Theorem 4.34 that $(\mu_i)_{x,\bar{x}}$, $i = 1, 2, \ldots, m$, are quasiconcave on $\mathbb{L}_X(x, \bar{x})$. In particular, we have

$$\mu_i(x) \leq \mu_i(z) \leq \mu_i(\bar{x}) = 1, \quad i = 1, 2, \ldots, m.$$

As A_m is nondecreasing in all arguments, we obtain

$$\begin{aligned} A_m(\mu_1(x), \ldots, \mu_m(x)) &\leq A_m(\mu_1(z), \ldots, \mu_m(z)) \\ &\leq A_m(\mu_1(\bar{x}), \ldots, \mu_m(\bar{x})). \end{aligned} \tag{5.17}$$

Applying $A_m(\mu_1(x), \ldots, \mu_m(x)) = \psi(x) \geq \alpha$ and using (5.17) we obtain that $\psi(z) = A_m(\mu_1(z), \ldots, \mu_m(z)) \geq \alpha$, implying $z = \lambda x + (1 - \lambda)\bar{x} \in U(\psi, \alpha)$ and also $\bar{x} \in U(\psi, \alpha)$. Consequently, $\bar{x} \in \mathrm{Ker}(U(\psi, \alpha))$, thus $U(\psi, \alpha)$ is starshaped. ■

The next example shows that the condition $\mathrm{Core}(\mu_1) \cap \cdots \cap \mathrm{Core}(\mu_m) \neq \emptyset$ is essential for validity of the result.

EXAMPLE 5.26 Let $X = \mathbf{R}$, let $\mu_i : \mathbf{R} \to [0,1]$, $i = 1, 2$, be defined as follows:

$$\mu_1(x) = \max\{0, 1 - x^2\}, \ \mu_2(x) = \max\{0, 1 - (x-1)^2\},$$

see Figure 5.2. Obviously, both μ_1 and μ_2 are quasiconcave on X, therefore, they are T_D-quasiconcave on X. We also have $\mathrm{Core}(\mu_1) = \{0\}$, $\mathrm{Core}(\mu_2) = \{1\}$, thus $\mathrm{Core}(\mu_1) \cap \mathrm{Core}(\mu_2) = \emptyset$. Let the aggregating mapping be the maximum, i.e., $A = S_M$. Then the function defined by $\psi(x) = A(\mu_1(x), \mu_2(x)) = \max\{\mu_1(x), \mu_2(x)\}$ is not quasiconcave on X; consequently, it is not upper-starshaped on X. Recall, that a function is upper-starshaped on $\mathbf{R}$ if and only if it is quasiconcave on $\mathbf{R}$. □

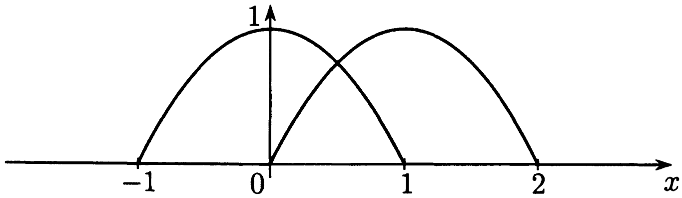

Figure 5.2.

PROPOSITION 5.27 *Let X be a convex subset of $\mathbf{R}^n$, and let $\mu_i : \mathbf{R}^n \to [0,1]$, $i = 1, 2, \ldots, m$, be upper normalized and T_D-quasiconcave on X and such that* $\mathrm{Core}(\mu_1) = \cdots = \mathrm{Core}(\mu_m) \neq \emptyset$. *Let $A_m : [0,1]^m \to [0,1]$ be a strictly monotone aggregating mapping. Then $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by*

$$\psi(x) = A_m(\mu_1(x), \ldots, \mu_m(x)) \tag{5.18}$$

is T_D-quasiconcave on X.

PROOF. Let $x, y \in X$ and $\lambda \in (0,1)$ be arbitrary. We have to prove that $\psi(\lambda x + (1-\lambda)y) \geq T_D(\psi(x), \psi(y))$. First, suppose $x \in \mathrm{Core}(\mu_i)$, $i = 1, 2, \ldots, m$. Then

$$\psi(x) = A_m(\mu_1(x), \ldots, \mu_m(x)) = A_m(1,1) = 1. \tag{5.19}$$

Letting $z = \lambda x + (1-\lambda)y$ then by T_D-quasiconcavity of μ_i we obtain

$$\mu_i(z) \geq T_D(\mu_i(x), \mu_i(y)) = \min\{1, \mu_i(y)\} = \mu_i(y), \quad i = 1, 2, \ldots, m.$$

By monotonicity of A_m we have

$$\psi(z) = A_m(\mu_1(z), \ldots, \mu_m(z)) \geq A_m(\mu_1(y), \ldots, \mu_m(y)) = \psi(y).$$

Applying (5.19) we get $T_D(\psi(x), \psi(y)) = T_D(1, \psi(y)) = \psi(y)$. Consequently, combining the previous two results, we obtain the required inequality $\psi(z) \geq T_D(\psi(x), \psi(y))$.

Second, suppose $x \notin \mathrm{Core}(\mu_i)$, $y \notin \mathrm{Core}(\mu_i)$, $i = 1, 2, \ldots, m$, then $\mu_i(x) < 1$, $\mu_i(y) < 1$, $i = 1, 2, \ldots, m$. By strict monotonicity of A_m we obtain $\psi(x) = A_m(\mu_1(x), \ldots, \mu_m(x)) < A_m(1,1) = 1$ and $\psi(y) = A_m(\mu_1(y), \mu_2(y)) < A_m(1,1) = 1$. Then by definition of T_D we get $T_D(\psi(x), \psi(y)) = 0$, hence, again $\psi(z) \geq T_D(\psi(x), \psi(y))$.

Since the role of x and y is symmetrical, the case $x \notin \mathrm{Core}(\mu_i)$, $y \in \mathrm{Core}(\mu_i)$, $i = 1, 2, \ldots, m$, has been already proved. ■

The above proposition allows for constructing new T_D-quasiconcave function on $X \subset \mathbf{R}^n$ from the original T_D-quasiconcave functions on $X \subset \mathbf{R}^n$ by using a strictly monotone aggregating operator, e.g., the t-conorm S_M.

The following example shows that the condition

$$\text{Core}(\mu_1) = \cdots = \text{Core}(\mu_m) \neq \emptyset$$

is essential for T_D-quasiconcavity of ψ defined by (5.18).

EXAMPLE 5.28 Let $X = \mathbf{R}^2$,

$$\mu_1(x, y) = \frac{1}{1+x^2},\ \mu_2(x, y) = \frac{1}{1+y^2},$$

$$A(a, b) = S_M(a, b) = \max\{a, b\},$$

and

$$\psi(x, y) = \max\{\mu_1(x, y), \mu_2(x, y)\} = \max\left\{\frac{1}{1+x^2}, \frac{1}{1+y^2}\right\},$$

see Figure 5.3. Obviously, μ_1 and μ_2 are quasiconcave on X. Therefore they are T-quasiconcave on X for every t-norm T. Setting $\breve{x} = 0$, $\breve{y} = 0$, we obtain $\mu_1(\breve{x}, \breve{y}) = \mu_2(\breve{x}, \breve{y}) = 1$. Evidently, the condition $\text{Core}(\mu_1) = \text{Core}(\mu_2)$ is not satisfied, even though $\text{Core}(\mu_1) \cap \text{Core}(\mu_2) \neq \emptyset$. Hence, by Proposition 5.25, ψ is upper-starshaped on X. Evidently, see Figure 5.3, ψ is not quasiconcave on X. Moreover, we show, that ψ is not T_D-quasiconcave on X. Suppose that ψ is T_D-quasiconcave on X. Setting $\bar{\mathbf{x}} = (\bar{x}, \bar{y}) = (0, 1)$, $\mathbf{x} = (x, y) = (1, 0)$, and $\psi_{\mathbf{x},\bar{\mathbf{x}}}(t) = \psi(t(\mathbf{x} - \bar{\mathbf{x}}) + \bar{\mathbf{x}})$, we obtain $\psi_{\mathbf{x},\bar{\mathbf{x}}}(0) = \psi_{\mathbf{x},\bar{\mathbf{x}}}(1) = 1$, and $\psi_{\mathbf{x},\bar{\mathbf{x}}}(\frac{1}{2}) = \frac{4}{5} < 1$, i.e., $\psi_{\mathbf{x},\bar{\mathbf{x}}}$ is not quasiconcave on $\mathbb{L}_X(\mathbf{x}, \bar{\mathbf{x}})$, a contradiction with Theorem 4.34. □

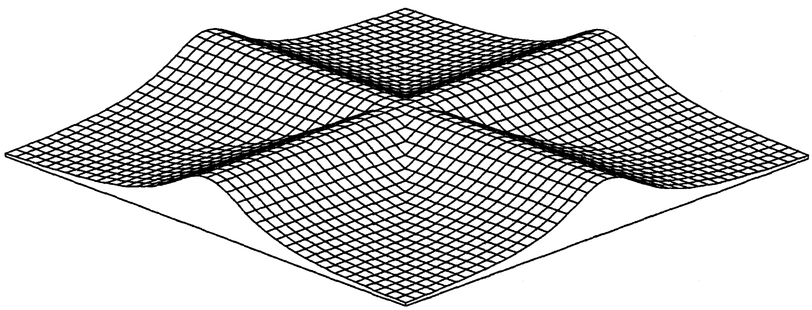

Figure 5.3.

EXAMPLE 5.29 Let $X = \mathbf{R}^2$ and let

$$\mu_1(x, y) = \max\{0, 1 - 10x^2 - y^2\},\ \mu_2(x, y) = \max\{0, 1 - x^2 - 10y^2\}.$$

Moreover, let

$$A(a,b) = \max\{a,b\} \text{ and } \psi(x,y) = A(\mu_1(x,y), \mu_2(x,y)).$$

Here, μ_1, μ_2 are quasiconcave on $\mathbf{R}^2$, thus T-quasiconcave on $\mathbf{R}^2$ for every t-norm T. Operator A is a strictly monotone aggregating operator. The condition $\text{Core}(\mu_1) = \text{Core}(\mu_2) = \{(0,0)\}$ is satisfied. By Proposition 5.27, ψ is T_D-quasiconcave on X, however, ψ is not quasiconcave on X, see Figure 5.4. □

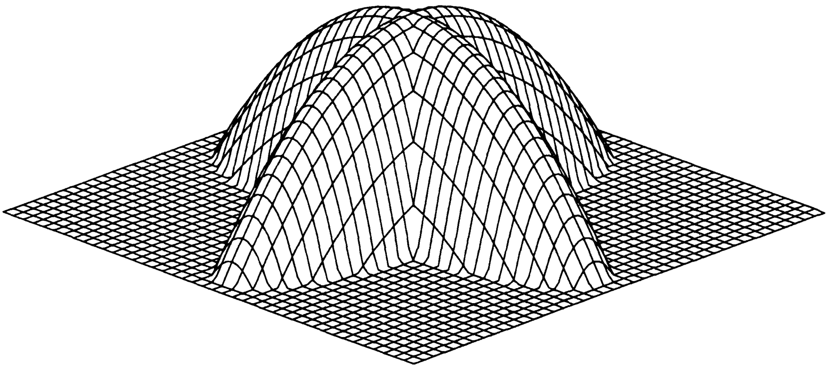

Figure 5.4.

The following definition extends the concept of domination between two triangular norms to aggregation operators.

DEFINITION 5.30 *An aggregation operator $A = \{A_n\}_{n=1}^{\infty}$ dominates an aggregation operator $A' = \{A'_n\}_{n=1}^{\infty}$, denoted by $A \gg A'$, if, for all $m \geq 2$ and all tuples $(x_1, x_2, \ldots, x_m) \in [0,1]^m$ and $(y_1, y_2, \ldots, y_m) \in [0,1]^m$, the following inequality holds*

$$A_m(A'_2(x_1,y_1), \ldots, A'_2(x_m,y_m)) \\ \geq A'_2(A_m(x_1,x_2,\ldots,x_m), A_m(y_1,y_2,\ldots,y_m)).$$

The following proposition generalizes Proposition 4.45.

PROPOSITION 5.31 *Let X be a convex subset of $\mathbf{R}^n$, let $A = \{A_n\}_{n=1}^{\infty}$ be an aggregation operator, T be a t-norm and let $\mu_i : \mathbf{R}^n \to [0,1], i = 1,2,\ldots,m$, be T-quasiconcave on X, and let A dominates T. Then $\varphi : \mathbf{R}^n \to [0,1]$ defined by*

$$\varphi(x) = A_m(\mu_1(x), \ldots, \mu_m(x)), \quad x \in \mathbf{R}^n,$$

is T-quasiconcave on X.

PROOF. As μ_i , $i = 1, 2, \ldots, m$, are T-quasiconcave on X, we have

$$\mu_i(\lambda x + (1-\lambda)y) \geq T(\mu_i(x), \mu_i(y))$$

for every $\lambda \in (0,1)$ and each $x, y \in X$. By monotonicity of aggregating mapping A_m, we obtain

$$\begin{aligned} &\varphi(\lambda x + (1-\lambda)y) \\ &\qquad = A_m(\mu_1(\lambda x + (1-\lambda)y), \ldots, \mu_m(\lambda x + (1-\lambda)y)) \\ &\qquad \geq A_m(T(\mu_1(x), \mu_1(y)), \ldots, T(\mu_m(x), \mu_m(y))). \end{aligned} \tag{5.20}$$

Using the fact that A dominates T, we obtain

$$\begin{aligned} &A_m(T(\mu_1(x), \mu_1(y)), \ldots, T(\mu_m(x), \mu_m(y))) \\ &\qquad \geq T(A_m(\mu_1(x), \ldots, \mu_m(x)), A_m(\mu_1(y), \ldots, \mu_m(y))) \\ &\qquad = T(\varphi(x), \varphi(y)), \end{aligned} \tag{5.21}$$

where $T = T^{(2)}$. Combining (5.20) and (5.21) we obtain the required result. ■

Chapter 6

FUZZY SETS

1. Introduction

A well known fact in the theory of sets is that properties of subsets of a given set X and their mutual relations can be studied by means of their characteristic functions, see, e.g., [11], [32] and [57]. While this may be advantageous in some contexts, we should notice that the notion of a characteristic function is more complex than the notion of a subset. Indeed, the characteristic function χ_A of a subset A of X is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Since χ_A is a function we need not only the underlying set X and its subset A but also one additional set, in this case the set $\{0, 1\}$ or any other two-element set. Moreover, we also need the notion of Cartesian product because functions are specially structured binary relations, in this case special subsets of $X \times \{0, 1\}$.

If we define fuzzy sets by means of their membership functions, that is, by replacing the range $\{0, 1\}$ of characteristic functions with a lattice, for example, the naturally ordered unit interval $[0, 1]$ of real numbers, then we should be aware of the following fact. Such functions may be related to certain objects (build from subsets of the underlying set) in an analogous way how the characteristic functions are related to subsets. This may explain why the fuzzy community (rightly?) hesitates to accept the view that a fuzzy subset of a given set is nothing else than its membership function. Then, a natural question arises. Namely, what are those objects? Obviously, it can be expected that they are more complex than just subsets because the class of functions mapping X into a lattice can be much richer than the class of characteristic functions. In the

next section, we show that it is advantageous to define these objects as nested families of subsets satisfying certain mild conditions.

Even if it is not the purpose of this chapter to deal with interpretations of the concepts involved, it should be noted that fuzzy sets and membership functions are closely related to the inherent imprecision of linguistic expressions in natural languages. Probability theory does not provide a way out, as it usually deals with crisp events and the uncertainty is whether this event will occur or not. However, in fuzzy logic, this is a matter of degree of truth rather than a simple "yes or no" decision.

Such generalized characteristic functions have found numerous connotations in different areas of mathematics, variety of philosophical interpretations and lot of real applications; see, e.g., [11], [29], [32], [57] and [75].

In the context of multicriteria decision making, functions mapping the underlying space into the unit interval $[0,1]$ and representing normalized utility functions can be also interpreted as membership functions of fuzzy sets of the underlying space, see [32].

The main purpose of this chapter is investigation of some properties of fuzzy sets, primarily with respect to generalized concave membership functions and with the prospect of applications in optimization and decision analysis.

2. Definition and Basic Properties

In order to define the concept of a fuzzy subset of a given set X within the framework of standard set theory we are motivated by the concept of upper level set of a function introduced in Chapter 2, see also [83] and [47]. Throughout this chapter, X is a nonempty set.

DEFINITION 6.1 *Let X be a nonempty set. A* fuzzy subset A of X *is the family of subsets $A_\alpha \subset X$, where $\alpha \in [0,1]$, satisfying the following properties:*

$$A_0 = X, \tag{6.1}$$

$$A_\beta \subset A_\alpha \quad \text{whenever } 0 \leq \alpha < \beta \leq 1, \tag{6.2}$$

$$A_\beta = \bigcap_{0 \leq \alpha < \beta} A_\alpha. \tag{6.3}$$

A fuzzy subset A of X will be also called a fuzzy set. *The class of all fuzzy subsets of X is denoted by $\mathcal{F}(X)$.*

DEFINITION 6.2 *Let $A = \{A_\alpha\}_{\alpha \in [0,1]}$ be a fuzzy subset of X. The $\mu_A : X \to [0,1]$ defined by*

$$\mu_A(x) = \sup\{\alpha \mid \alpha \in [0,1],\, x \in A_\alpha\} \tag{6.4}$$

is called the membership function of A, and the value $\mu_A(x)$ is called membership degree of x in the fuzzy set A.

DEFINITION 6.3 *Let A be a fuzzy subset of X. The core of A, $\mathrm{Core}(A)$, is defined by*

$$\mathrm{Core}(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

If the core of A is nonempty, then A is said to be normalized. The support of A, $\mathrm{Supp}(A)$, is defined by

$$\mathrm{Supp}(A) = \mathrm{Cl}(\{x \in X \mid \mu_A(x) > 0\}).$$

The height of A, $\mathrm{Hgt}(A)$, is defined by

$$\mathrm{Hgt}(A) = \sup\{\mu_A(x) \mid x \in X\}.$$

The upper-level set of the membership function μ_A of A at $\alpha \in [0,1]$ is denoted by $[A]_\alpha$ and called the α-cut of A, that is,

$$[A]_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}. \tag{6.5}$$

Note that the core and support of A coincide with the core and support of μ_A introduced in Definition 4.26. Also note that if A is normalized, then $\mathrm{Hgt}(A) = 1$, but not vice versa.

In the following two propositions, we show that the family generated by the upper level sets of a function $\mu : X \longrightarrow [0,1]$, satisfies conditions (6.1) - (6.3), thus, it generates a fuzzy subset of X and the membership function μ_A defined by (6.4) coincides with μ. Moreover, for a given fuzzy set $A = \{A_\alpha\}_{\alpha\in[0,1]}$, every α-cut $[A]_\alpha$ given by (6.5) coincides with the corresponding A_α.

PROPOSITION 6.4 *Let $\mu : X \longrightarrow [0,1]$ be a function and let $A = \{A_\alpha\}_{\alpha\in[0,1]}$ be a family of its upper-level sets, i.e. $A_\alpha = U(\mu, \alpha)$ for all $\alpha \in [0,1]$. Then A is a fuzzy subset of X and μ is the membership function of A.*

PROOF. First, we prove that $A = \{A_\alpha\}_{\alpha\in[0,1]}$, where $A_\alpha = U(\mu, \alpha)$ for all $\alpha \in [0,1]$ satisfies conditions (6.1) - (6.3). Indeed, conditions (6.1) and (6.2) hold easily. For condition (6.3), we observe that by (6.2) it follows that $A_\beta \subset \bigcap\limits_{0\leq\alpha<\beta} A_\alpha$. To prove the opposite inclusion, let

$$x \in \bigcap_{0\leq\alpha<\beta} A_\alpha. \tag{6.6}$$

Assume the contrary, that is, let $x \notin A_\beta$. Then $\mu(x) < \beta$ and there exists α' such that $\mu(x) < \alpha' < \beta$. By (6.6) we have $x \in A_{\alpha'}$, thus $\mu(x) \geq \alpha'$, a contradiction.

It remains to prove that $\mu = \mu_A$, where μ_A is the membership function of A. For this purpose let $x \in X$ and let us show that $\mu(x) = \mu_A(x)$.

By definition (6.4) we have

$$\begin{aligned}\mu_A(x) &= \sup\{\alpha \mid \alpha \in [0,1], x \in A_\alpha\}\\ &= \sup\{\alpha \mid \alpha \in [0,1], x \in U(\mu,\alpha)\}\\ &= \sup\{\alpha \mid \alpha \in [0,1], \mu(x) \geq \alpha\},\end{aligned}$$

therefore, $\mu(x) = \mu_A(x)$. ■

PROPOSITION 6.5 *Let $A = \{A_\alpha\}_{\alpha\in[0,1]}$ be a fuzzy subset of X and let $\mu_A : X \longrightarrow [0,1]$ be the membership function of A. Then for each $\alpha \in [0,1]$ the α-cut $[A]_\alpha$ is equal to A_α.*

PROOF. Let $\beta \in [0,1]$. By definition (6.4), observe that $A_\beta \subset [A]_\beta$. It suffices to prove the opposite inclusion.

Let $x \in [A]_\beta$. Then $\mu_A(x) \geq \beta$, or, equivalently,

$$\sup\{\alpha \mid \alpha \in [0,1], x \in A_\alpha\} \geq \beta.$$

It follows that for every β' with $\beta' < \beta$ there exists α' with $\beta' \leq \alpha' \leq \beta$, such that $x \in A_{\alpha'}$. By monotonicity condition (6.2) we have $x \in A_\alpha$ for all $0 \leq \alpha \leq \alpha'$. Hence, $x \in \bigcap_{0\leq\alpha<\beta} A_\alpha$, however, applying (6.3) we get $x \in A_\beta$. Consequently, $[A]_\beta \subset A_\beta$. ■

These results allow for introducing a natural one-to-one correspondence between fuzzy subsets of X and real-valued functions mapping X to $[0,1]$. Any fuzzy subset A of X is given by its membership function μ_A and vice-versa, any function $\mu : X \longrightarrow [0,1]$ uniquely determines a fuzzy subset A of X, with the property that the membership function μ_A of A is μ.

The notions of inclusion and equality extend to fuzzy subsets as follows. Let $A = \{A_\alpha\}_{\alpha\in[0,1]}$, $B = \{B_\alpha\}_{\alpha\in[0,1]}$ be fuzzy subsets of X. Then

$$A \subset B \quad \text{if} \quad A_\alpha \subset B_\alpha \quad \text{for each } \alpha \in [0,1], \tag{6.7}$$

$$A = B \quad \text{if} \quad A_\alpha = B_\alpha \quad \text{for each } \alpha \in [0,1]. \tag{6.8}$$

PROPOSITION 6.6 *Let $A = \{A_\alpha\}_{\alpha\in[0,1]}$ and $B = \{B_\alpha\}_{\alpha\in[0,1]}$ be fuzzy subsets of X. Then the following holds:*

$$A \subset B \quad \textit{if and only if} \quad \mu_A(x) \leq \mu_B(x) \quad \textit{for all } x \in X, \tag{6.9}$$

$$A = B \quad \textit{if and only if} \quad \mu_A(x) = \mu_B(x) \quad \textit{for all } x \in X. \tag{6.10}$$

PROOF. We prove only (6.9), the proof of statement (6.10) is analogical. Let $x \in X$, $A \subset B$. Then by definition (6.7), $A_\alpha \subset B_\alpha$ for each $\alpha \in [0,1]$.

Using (6.4) we obtain

$$\begin{aligned}\mu_A(x) &= \sup\{\alpha \mid \alpha \in [0,1],\, x \in A_\alpha\} \\ &\leq \sup\{\alpha \mid \alpha \in [0,1],\, x \in B_\alpha\} \\ &= \mu_B(x).\end{aligned}$$

Suppose that $\mu_A(x) \leq \mu_B(x)$ holds for all $x \in X$ and let $\alpha \in [0,1]$. We have to show that $A_\alpha \subset B_\alpha$. Indeed, for an arbitrary $u \in A_\alpha$, we have

$$\sup\{\beta \mid \beta \in [0,1],\, u \in A_\beta\} \leq \sup\{\beta \mid \beta \in [0,1],\, u \in B_\beta\}.$$

From here, $\sup\{\beta \mid \beta \in [0,1],\, u \in B_\beta\} \geq \alpha$, therefore, for each $\beta < \alpha$, it follows that $u \in B_\beta$. Hence, by (6.1) in Definition 6.1, we obtain $u \in B_\alpha$. ∎

A subset of X can be considered as a special fuzzy subset of X where all members of its defining a family consist of the same elements. This is formalized in the following definition.

DEFINITION 6.7 *Let A be a subset of X. The fuzzy subset $\{A_\alpha\}_{\alpha\in[0,1]}$ of X defined by $A_\alpha = A$ for all $\alpha \in (0,1]$ is called a* crisp fuzzy subset of X generated by A. *A fuzzy subset of X generated by some $A \subset X$ is called a* crisp fuzzy subset of X *or briefly a crisp subset of X.*

PROPOSITION 6.8 *Let $\{A\}_{\alpha\in[0,1]}$ be a crisp subset of X generated by A. Then the membership function of $\{A\}_{\alpha\in[0,1]}$ is equal to the characteristic function of A.*

PROOF. Let μ be the membership function of $\{A\}_{\alpha\in[0,1]}$. We wish to prove that

$$\mu(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Let $x \in X, x \notin A$. Then by definition (6.4) and Definition 6.7,

$$\mu(x) = \sup\{\alpha \mid \alpha \in [0,1],\, x \in A_\alpha\} = 0.$$

Let $x \in X, x \in A$. Then again by definition (6.4) and Definition 6.7,

$$\begin{aligned}\mu(x) &= \sup\{\alpha \mid \alpha \in [0,1],\, x \in A_\alpha\} \\ &= \sup\{\alpha \mid \alpha \in [0,1],\, x \in A\} \\ &= 1.\end{aligned}$$

∎

By Definition 6.7, the set $\mathcal{P}(X)$ of all subsets of X can naturally be embedded into the set of all fuzzy subsets of X and we can write $A = \{A_\alpha\}_{\alpha\in[0,1]}$ if $\{A_\alpha\}_{\alpha\in[0,1]}$ is generated by $A \subset X$. According to Proposition 6.8, we have in

this case $\mu_A = \chi_A$. In particular, if A contains only one element a of X, that is, $A = \{a\}$, then we write $a \in \mathcal{F}(X)$ instead of $\{a\} \in \mathcal{F}(X)$ and χ_a instead of $\chi_{\{a\}}$.

EXAMPLE 6.9 Let $\mu : \mathbf{R} \to [0,1]$ be defined by $\mu(x) = e^{-x^2}$. Let $A' = \{A'_\alpha\}_{\alpha\in[0,1]}$, $A'' = \{A''_\alpha\}_{\alpha\in[0,1]}$ be two families of subsets in $\mathbf{R}$ defined as follows:

$$\begin{aligned} A'_\alpha &= \{x \mid x \in \mathbf{R},\ \mu(x) > \alpha\}, \\ A''_\alpha &= \{x \mid x \in \mathbf{R},\ \mu(x) \geq \alpha\}. \end{aligned}$$

Clearly, A'' is a fuzzy subset of $\mathbf{R}$ and $A' \neq A''$. Observe that (6.1) and (6.2) are satisfied for A' and A''. However, $A'_1 = \emptyset$ and $\bigcap_{0\leq\alpha<1} A'_\alpha = \{0\}$, thus (6.3) is not satisfied. Hence A' is not a fuzzy subset of $\mathbf{R}$. □

3. Operations with Fuzzy Sets

In order to generalize the set operations of intersection, union and complement to fuzzy set operations, it is natural to use triangular norms, triangular conorms and fuzzy negations introduced in Chapter 4.

Given a De Morgan triple (T, S, N), i.e., a t-norm T, a t-conorm S and a fuzzy negation N, introduced in Definition 4.21, we define the operations *intersection* $\cap_T$, *union* $\cup_S$ and *complement* $\mathcal{C}_N$ *on* $\mathcal{F}(X)$ as follows: Let A and B be fuzzy subsets of X, and μ_A and μ_B be their membership functions. Then the membership functions of the fuzzy subsets $A \cap_T B$, $A \cup_S B$ and $\mathcal{C}_N A$ of X are defined by

$$\begin{aligned} \mu_{A\cap_T B}(x) &= T(\mu_A(x), \mu_B(x)), \\ \mu_{A\cup_S B}(x) &= S(\mu_A(x), \mu_B(x)), \\ \mu_{\mathcal{C}_N A}(x) &= N(\mu_A(x)). \end{aligned}$$

The operations introduced by L. Zadeh in [57] have been originally based on $T = T_M = \min$, $S = S_M = \max$ and standard negation N defined in (4.20). The properties of the operations intersection $\cap_T$, union $\cup_S$ and complement $\mathcal{C}_N$ can be derived directly from the corresponding properties of t-norm T, t-conorm S and fuzzy negation N. For brevity, in case of $T = \min$ and $S = \max$, we write only $\cap$ and $\cup$, instead of $\cap_T$ and $\cup_S$.

Notice that for $A \in \mathcal{F}(X)$ we do not necessarily obtain properties which hold for subsets of X. For example,

$$A \cap_T \mathcal{C}_N A = \emptyset, \tag{6.11}$$
$$A \cup_S \mathcal{C}_N A = X, \tag{6.12}$$

may not hold. If the t-norm T in the De Morgan triple (T, S, N) does not have zero divisors, e.g., $T = \min$, then these properties never hold unless A is a

crisp set. On the other hand, for the De Morgan triple (T_L, S_L, N) based on Łukaszewicz t-norm $T = T_L$, properties (6.11) and (6.12) are satisfied.

Given a t-norm T and fuzzy subsets A and B of X and Y, respectively, the *Cartesian product* $A \times_T B$ is the fuzzy subset of $X \times Y$ with the following membership function:

$$\mu_{A\times_T B}(x,y) = T(\mu_A(x), \mu_B(y)) \quad \text{for } (x,y) \in X \times Y. \tag{6.13}$$

An interesting and natural question arises, whether the α-cuts of the intersection $A \cap_T B$, union $A \cup_S B$ and Cartesian product $A \times_T B$ of $A, B \in \mathcal{F}(X)$, coincide with the intersection, union and Cartesian product, respectively, of the corresponding α-cuts $[A]_\alpha$ and $[B]_\alpha$. We have the following result, see [57].

PROPOSITION 6.10 *Let T be a t-norm, S be a t-conorm, $\alpha \in [0,1]$. Then the equalities*

$$\begin{aligned} [A \cap_T B]_\alpha &= [A]_\alpha \cap [B]_\alpha, \\ [A \cup_S B]_\alpha &= [A]_\alpha \cup [B]_\alpha, \\ [A \times_T B]_\alpha &= [A]_\alpha \times [B]_\alpha, \end{aligned} \tag{6.14}$$

hold for all fuzzy sets $A, B \in \mathcal{F}(X)$ if and only if α is an idempotent element of both T and S.

In particular, equalities (6.14) hold for all $\alpha \in [0,1]$ and for all fuzzy sets $A, B \in \mathcal{F}(X)$ if and only if $T = T_M$ and $S = S_M$.

4. Extension Principle

The purpose of the extension principle proposed by L. Zadeh in [136] and [137] is to extend functions or operations having crisp arguments to functions or operations with fuzzy set arguments. Zadeh's methodology can be cast in a more general setting of carrying a membership function via a mapping, see, e.g., [29]. There exist other generalizations for set-to-set mappings; see, e.g., [29], [85]. From now on, X and Y are nonempty sets.

DEFINITION 6.11 (Extension Principle) *Let X, Y be sets, $f : X \to Y$ be a mapping. The mapping $\tilde{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$ defined for all $A \in \mathcal{F}(X)$ with $\mu_A : X \to [0,1]$ and all $y \in Y$ by*

$$\mu_{\tilde{f}(A)}(y) = \begin{cases} \sup\{\mu_A(x) \mid x \in X,\, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases} \tag{6.15}$$

is called a fuzzy extension of f.

By formula (6.15) we define the membership function of the image of the fuzzy set A by fuzzy extension $\tilde{f}$. A justification of this concept is given in

the following theorem stating that the mapping $\tilde{f}$ is a true extension of the mapping f when considering the natural embedding of $\mathcal{P}(X)$ into $\mathcal{F}(X)$ and $\mathcal{P}(Y)$ into $\mathcal{F}(Y)$.

PROPOSITION 6.12 *Let X, Y be sets, $f : X \to Y$ be a mapping, $x_0 \in X$, $y_0 = f(x_0)$. If $\tilde{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$ is defined by (6.15), then*

$$\tilde{f}(x_0) = y_0,$$

and the membership function $\mu_{\tilde{f}(x_0)}$ of the fuzzy set $\tilde{f}(x_0)$ is a characteristic function of y_0, i.e.

$$\mu_{\tilde{f}(x_0)} = \chi_{y_0}. \tag{6.16}$$

PROOF. To prove the theorem, it is sufficient to prove (6.16). Remember that we identify subsets and points of X and Y with the corresponding crisp fuzzy subsets.

Let $y \in Y$, we will show that

$$\mu_{\tilde{f}(x_0)}(y) = \chi_{y_0}(y). \tag{6.17}$$

Let $y = y_0$. Since $y_0 = f(x_0)$ we obtain by (6.15) that $\mu_{\tilde{f}(x_0)}(y) = \chi_{x_0}(x_0) = 1$. Moreover, by the definition of characteristic function we have $\chi_{y_0}(y_0) = 1$, thus (6.17) is satisfied.

On the other hand, let $y \neq y_0$. Again, by the definition of characteristic function we have $\chi_{y_0}(y) = 0$. As $y \neq f(x_0)$ we obtain for all $x \in X$ with $y = f(x)$ that $x \neq x_0$. Clearly, $\chi_{x_0}(x) = 0$ and by (6.15) it follows that $\mu_{\tilde{f}(x_0)}(y) = 0$, which was required. ■

A more general form of Proposition 6.12 says that the image of a crisp set by a fuzzy extension of a function is again crisp.

THEOREM 6.13 *Let X, Y be sets, $f : X \to Y$ be a mapping, $A \subset X$. Then*

$$\tilde{f}(A) = f(A)$$

and the membership function $\mu_{\tilde{f}(A)}$ of $\tilde{f}(A)$ is a characteristic function of the set $f(A)$, i.e.

$$\mu_{\tilde{f}(A)} = \chi_{f(A)}. \tag{6.18}$$

PROOF. We prove only (6.18). Let $y \in Y$. Since A is crisp, $\mu_A = \chi_A$. By (6.15) we obtain

$$\begin{aligned} \mu_{\tilde{f}(A)} &= \max\{0, \sup\{\chi_A(t) \mid t \in X,\ f(t) = y\}\} \\ &= \begin{cases} 1 & \text{if } y \in f(A), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Consequently, $\mu_{\tilde{f}(A)}(y) = \chi_{f(A)}(y)$. ■

In the following sections the extension principle will be used in different settings for various sets X and Y, and also for different classes of mappings.

The mathematics of fuzzy sets is, in a narrow sense, a mathematics of the space of membership functions. In this chapter we deal with some properties of this space related to the set of (generalized) quasiconcave membership functions.

5. Binary and Valued Relations

In the classical set theory, a *binary relation* R between the elements of sets X and Y is defined as a subset of the Cartesian product $X \times Y$, that is, $R \subset X \times Y$. A valued relation on $X \times Y$ will be a fuzzy subset of $X \times Y$.

DEFINITION 6.14 *A* valued relation R on $X \times Y$ *is a fuzzy subset of* $X \times Y$. *The set of all valued relations on* $X \times Y$ *is denoted by* $\mathcal{F}(X \times Y)$.

The valued relations are sometimes called fuzzy relations, however, we reserve this name for valued relations defined on $\mathcal{F}(X) \times \mathcal{F}(Y)$, which will be defined later.

Every binary relation R, where $R \subset X \times Y$, is embedded into the class of valued relations on $X \times Y$ by its characteristic function χ_R being understood as its membership function μ_R. In this sense, any binary relation is valued.

Particularly, any function $f : X \to Y$ is considered as a binary relation, that is, as a subset R_f of $X \times Y$, where

$$R_f = \{(x, y) \in X \times Y \mid y = f(x)\}. \tag{6.19}$$

Here, R_f may be identified with the valued relation by its characteristic function

$$\mu_{R_f}(x, y) = \chi_{R_f}(x, y) \tag{6.20}$$

for all $(x, y) \in X \times Y$, where

$$\chi_{R_f}(x, y) = \chi_{f(x)}(y). \tag{6.21}$$

In particular, if $Y = X$, then each valued relation R on $X \times X$ is a fuzzy subset of $X \times X$, and it is called a valued relation on X instead of on $X \times X$.

DEFINITION 6.15 *Let T be a triangular norm. A valued relation R on X is*

(i) reflexive *if for each* $x \in X$

$$\mu_R(x, x) = 1;$$

(ii) symmetric *if for each* $x, y \in X$

$$\mu_R(x, y) = \mu_R(y, x);$$

(iii) T-transitive *if for each* $x, y, z \in X$

$$T(\mu_R(x, y), \mu_R(y, z)) \leq \mu_R(x, z);$$

(iv) separable *if*

$$\mu_R(x, y) = 1 \text{ if and only if } x = y;$$

(v) T-equivalence *if R is reflexive, symmetric and T-transitive;*

(vi) T-equality *if R is reflexive, symmetric, T-transitive and separable.*

DEFINITION 6.16 *Let R be a valued relation on $X \times Y$ and let $N : [0, 1] \to [0, 1]$ be a negation.*

(i) *A valued relation R^{-1} on $Y \times X$ is the inverse of R if $\mu_{R^{-1}}(y, x) = \mu_R(x, y)$ for each $x \in X$ and $y \in Y$.*

(ii) *A valued relation $C_N R$ on $X \times Y$ is the complement of R if $\mu_{C_N R}(x, y) = N(\mu_R(x, y))$ for each $x \in X$ and $y \in Y$. If N is the standard negation, then the index N is omitted.*

(iii) *If μ_R is upper semicontinuous on $X \times Y$, then R is called closed.*

For more information about valued relations, see [32].

EXAMPLE 6.17 Let $\varphi : \mathbf{R} \to [0, 1]$ be a function. Then R defined by the membership function μ_R for all $x, y \in \mathbf{R}$ by

$$\mu_R(x, y) = \varphi(x - y) \tag{6.22}$$

is a valued relation on $\mathbf{R}$. If

$$\varphi(t) = \begin{cases} 1 & \text{if } t \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

then R defined by (6.22) is the usual binary relation $\leq$ on $\mathbf{R}$. If

$$\varphi(t) = \begin{cases} 1 & \text{if } t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

then R defined by (6.22) is the usual binary relation $\geq$ on $\mathbf{R}$. If

$$\varphi(t) = \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{otherwise,} \end{cases}$$

then R defined by (6.22) is the usual binary relation $=$ on $\mathbf{R}$. □

6. Fuzzy Relations

Let X, Y be nonempty sets. Consider a valued relation R on $X \times Y$ given by the membership function $\mu_R : X \times Y \to [0,1]$. In order to extend this function with crisp arguments to function with fuzzy arguments, we apply the extension principle (6.42) in Definition 6.11. Then we obtain a mapping $\tilde{\mu}_R : \mathcal{F}(X \times Y) \to \mathcal{F}([0,1])$, that is, values of $\tilde{\mu}_R$ are fuzzy subsets of $[0,1]$.

Since $\mathcal{F}([0,1])$ can be considered as a lattice, we can consider $\tilde{\mu}_R$ as the membership function of an L-fuzzy set.

However, we do not follow this way, instead, we follow a more practical way and define fuzzy relations as valued relations on $\mathcal{F}(X) \times \mathcal{F}(Y)$.

DEFINITION 6.18 *A fuzzy subset of $\mathcal{F}(X) \times \mathcal{F}(Y)$ is called a* fuzzy relation on $\mathcal{F}(X) \times \mathcal{F}(Y)$. *The set of all fuzzy relations on $\mathcal{F}(X) \times \mathcal{F}(Y)$ is denoted by $\mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(Y))$.*

Further on, we shall investigate mappings Ψ assigning to each valued relation R from $\mathcal{F}(X \times Y)$ a fuzzy relation from $\mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(Y))$, that is,

$$\Psi : \mathcal{F}(X \times Y) \to \mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(Y)).$$

DEFINITION 6.19 *Let R be a valued relation on $X \times Y$. A fuzzy relation $\tilde{R}$ on $\mathcal{F}(X) \times \mathcal{F}(Y)$ given by the membership function $\mu_{\tilde{R}} : \mathcal{F}(X) \times \mathcal{F}(Y) \to [0,1]$ is called a* fuzzy extension of relation R, *if, for each $x \in X, y \in Y$, it holds*

$$\mu_{\tilde{R}}(x,y) = \mu_R(x,y). \qquad (6.23)$$

DEFINITION 6.20 *Let $\Psi : \mathcal{F}(X \times Y) \to \mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(Y))$ be a mapping. Let for all $R \in \mathcal{F}(X \times Y)$, $\Psi(R)$ be a fuzzy extension of relation R. Then the mapping Ψ is called a* fuzzy extension of valued relations.

DEFINITION 6.21 *Let $\Phi, \Psi : \mathcal{F}(X \times Y) \to \mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(Y))$ be mappings. We say that the* mapping Φ is dual to Ψ, *if*

$$\Phi(CR) = C\Psi(R) \qquad (6.24)$$

holds for all $R \in \mathcal{F}(X \times Y)$. For Φ dual to Ψ, $R \in \mathcal{F}(X \times Y)$, the fuzzy relation $\Phi(R)$ is called dual *to fuzzy relation $\Psi(R)$.*

Notice that a mapping Φ is dual to Ψ, if and only if the mapping Ψ is dual to Φ. This fact follows from (6.24) and from the identity

$$CCR = R.$$

The analogical statement holds for the dual fuzzy relations $\Phi(R)$ and $\Psi(R)$.

Now, we define an important special fuzzy extension mapping of a valued relations.

DEFINITION 6.22 *Let T be a t-norm. A mapping $\Psi^T : \mathcal{F}(X \times Y) \to \mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(Y))$ defined for every valued relation $R \in \mathcal{F}(X \times Y)$ and for all fuzzy sets A, B with the membership functions $\mu_A : X \to [0,1]$, $\mu_B : Y \to [0,1]$, respectively, by*

$$\mu_{\Psi^T(R)}(A, B) = \sup\{T(\mu_R(x, y), T(\mu_A(x), \mu_B(y))) \mid x \in X,\ y \in Y\}, \tag{6.25}$$

is called a T-fuzzy extension of valued relations. *Given $R \in \mathcal{F}(X \times Y)$, then the fuzzy relation $\Psi^T(R)$ is called the* T-fuzzy extension of relation R.

In the following proposition we show that the T-fuzzy extension Ψ^T of valued relations is a fuzzy extension of valued relations in the sense of Definition 6.19.

PROPOSITION 6.23 *Let T be a t-norm. Let R be a valued relation on $X \times Y$. If $\tilde{R} = \Psi^T(R)$ is the T-fuzzy extension of relation R, then $\tilde{R}$ is a fuzzy extension of relation R. Moreover, if $A', A'' \in \mathcal{F}(X), B', B'' \in \mathcal{F}(Y)$ and*

$$A' \subset A'',\ B' \subset B'',$$

then

$$\mu_{\tilde{R}}(A', B') \leq \mu_{\tilde{R}}(A'', B''). \tag{6.26}$$

PROOF. Let $x \in X, y \in Y$. By (6.25) we obtain

$$\begin{aligned} \mu_{\Psi^T(R)}(x, y) &= \sup\{T(\mu_R(u, v), T(\chi_x(u), \chi_y(\nu)) \mid u \in X,\ v \in Y\} \\ &= T(\mu_R(x, y), T(1, 1)) = \mu_R(x, y). \end{aligned}$$

Observe that, for all $u \in X, v \in Y$, the inequalities $\mu_{A'}(u) \leq \mu_{A''}(u)$ and $\mu_{B'}(v) \leq \mu_{B''}(v)$ hold. Clearly, (6.26) follows from the monotonicity of the t-norm T. ∎

EXAMPLE 6.24 Let $f : X \to Y$ be a mapping, let the corresponding relation R_f be defined by (6.19) and (6.20). Let T be a t-norm, let A and B be fuzzy subsets of X and Y given by the membership functions $\mu_A : X \to [0,1]$, $\mu_B : Y \to [0,1]$, respectively. Let $y \in Y$ and let B be defined for all $z \in Y$ as follows: $\mu_B(z) = \chi_y(z)$. Then by (6.25) we get the T-fuzzy extension $\Psi^T(R_f)$ of relation R_f as

$$\begin{aligned} &\mu_{\Psi^T(R_f)}(A, B) \\ &\quad = \max\{0, \sup\{T(\mu_{R_f}(x, z), T(\mu_A(x), \chi_y(z)) \mid x \in X,\ z \in Y\}\}. \end{aligned} \tag{6.27}$$

The value $\mu_{\Psi^T(R_f)}(A, B)$ expresses the degree in which $y \in Y$ is considered as the image of $A \in \mathcal{F}(X)$ through the mapping f. □

The following proposition says that extension principle (6.15) is a special t-norm independent fuzzy extension of relation (6.19).

PROPOSITION 6.25 *Let $f : X \longrightarrow Y$ be a mapping, let the corresponding relation R_f be defined by (6.19) and (6.20). Let T be a t-norm and A, B be fuzzy subsets with the corresponding membership functions $\mu_A : X \longrightarrow [0, 1]$, $\mu_B : Y \longrightarrow [0, 1]$, respectively. Let $y \in Y$ and let μ_B be defined for all $z \in Y$ by $\mu_B(z) = \chi_y(z)$. Then, for the membership function of the T-fuzzy extension $\Psi^T(R_f)$ of relation R_f,*

$$\mu_{\Psi^T(R_f)}(A, B) = \mu_{\tilde{f}(A)}(y),$$

where $\mu_{\tilde{f}(A)}(y)$ is defined by (6.15).

PROOF. Let $x \in f^{-1}(y)$. For $z = y$, we get

$$T(\mu_A(x), \chi_y(z)) = T(\mu_A(x), 1) = \mu_A(x)$$

and, by (6.20) and (6.21), $\mu_{R_f}(x, y) = \chi_{f(x)}(y) = 1$. It follows from (6.27) that

$$\begin{aligned} \mu_{\Psi^T(R_f)}(A, B) &= \sup\{T(1, \mu_A(x)) \mid x \in X, \ f(x) = y\} \\ &= \sup\{\mu_A(x) \mid x \in X, \ f(x) = y\} \\ &= \mu_{\tilde{f}(A)}(y). \end{aligned}$$

Next, if $f^{-1}(y) = \emptyset$, then $\mu_{R_f}(x, z) = 0$ for all $x \in X$, $z \in Y$. By (6.27) we obtain

$$\begin{aligned} \mu_{\Psi^T(R_f)}(A, B) &= \sup\{T(\mu_{R_f}(x, z), T(\mu_A(x), \chi_y(z))) \mid x \in X, \ z \in Y\} \\ &= \sup\{T(0, T(\mu_A(x), \chi_y(z))) \mid x \in X, \ z \in Y\} \\ &= 0. \end{aligned}$$

However, by (6.15), $\mu_{\tilde{f}(A)}(y) = 0$. ■

In the following section we shall introduce another fuzzy extensions of valued relations.

7. Fuzzy Extensions of Valued Relations

In the preceding section, Definition 6.22, we have introduced a T-fuzzy extension $\Psi^T(R)$ of a valued relation R, where T has been a t-norm. For arbitrary fuzzy sets A, B with the membership functions $\mu_A : X \longrightarrow [0, 1]$, μ_B :

$Y \to [0,1]$, respectively, the T-fuzzy extension $\Psi^T(R)$ of a valued relation R has been defined by (6.25). The T-fuzzy extension of valued relations is the most common mapping used in applications. However, in possibility theory the other mappings based on t-norms and t-conorms are well known; see, e.g., [41].

Let X, Y be nonempty sets. In the following definition we introduce six fuzzy extensions of valued relations, including the previously defined T-fuzzy extension. Later, in Part II, these mappings and the corresponding fuzzy relations will be used for comparing left and right sides of the constraints in mathematical programming problems.

DEFINITION 6.26 *Let T be a t-norm, S be a t-conorm.*

(i) *A mapping $\Psi^T : \mathcal{F}(X \times Y) \to \mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(Y))$ is defined for every valued relation $R \in \mathcal{F}(X \times Y)$ and for all fuzzy sets A, B with the membership functions $\mu_A : X \to [0,1]$, $\mu_B : Y \to [0,1]$, respectively, by*

$$\mu_{\Psi^T(R)}(A, B) = \sup\{T(T(\mu_A(x), \mu_B(y)), \mu_R(x, y)) \mid x \in X,\ y \in Y\}. \tag{6.28}$$

(ii) *A mapping $\Psi_S : \mathcal{F}(X \times Y) \to \mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(Y))$ is defined for every valued relation $R \in \mathcal{F}(X \times Y)$ and for all fuzzy sets $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$ by*

$$\mu_{\Psi_S(R)}(A, B) = \inf\{S(S(\mu_{CA}(x), \mu_{CB}(y)), \mu_R(x, y)) \mid x \in X,\ y \in Y\}. \tag{6.29}$$

(iii) *A mapping $\Psi^{T,S} : \mathcal{F}(X \times Y) \to \mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(Y))$ is defined for every valued relation $R \in \mathcal{F}(X \times Y)$ and for all fuzzy sets $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$ by*

$$\mu_{\Psi^{T,S}(R)}(A, B) = \sup\{\inf\{T(\mu_A(x), S(\mu_{CB}(y), \mu_R(x, y))) \mid y \in Y\} \mid x \in X\}. \tag{6.30}$$

(iv) *A mapping $\Psi_{T,S} : \mathcal{F}(X \times Y) \to \mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(Y))$ is defined for every valued relation $R \in \mathcal{F}(X \times Y)$ and for all fuzzy sets $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$ by*

$$\mu_{\Psi_{T,S}(R)}(A, B) = \inf\{\sup\{S(T(\mu_A(x), \mu_R(x, y)), \mu_{CB}(y)) \mid x \in X\} \mid y \in Y\}. \tag{6.31}$$

(v) *A mapping* $\Psi^{S,T} : \mathcal{F}(X \times Y) \to \mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(Y))$ *is defined for every valued relation* $R \in \mathcal{F}(X \times Y)$ *and for all fuzzy sets* $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$ *by*

$$\mu_{\Psi^{S,T}(R)}(A, B) \\ = \sup\{\inf\{T(S(\mu_{CA}(x), \mu_R(x,y)), \mu_B(y)) \mid x \in X\} \mid y \in Y\} \tag{6.32}$$

(vi) *A mapping* $\Psi_{S,T} : \mathcal{F}(X \times Y) \to \mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(Y))$ *is defined for every valued relation* $R \in \mathcal{F}(X \times Y)$ *and for all fuzzy sets* $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$ *by*

$$\mu_{\Psi_{S,T}(R)}(A, B) \\ = \inf\{\sup\{S(\mu_{CA}(x), T(\mu_B(y), \mu_R(x,y))) \mid y \in Y\} \mid x \in X\}. \tag{6.33}$$

Now, we show that all mappings defined in Definition 6.26 are fuzzy extensions of valued relations in the sense of Definition 6.19.

PROPOSITION 6.27 *Let T be a t-norm, S be a t-conorm. Then the mappings*

$$\Psi^T,\ \Psi_S,\ \Psi^{T,S},\ \Psi_{T,S},\ \Psi^{S,T},\ \Psi_{S,T}, \tag{6.34}$$

defined by (6.28) - (6.33), are fuzzy extensions of valued relations according to Definition 6.19.

PROOF. The statement (i) has been proved already in Proposition 6.23. The other statements can be proved analogously. We omit the detailed proofs. ■

PROPOSITION 6.28 *Let* R *be a binary relation on* $X \times Y$, A *and* B *be nonempty crisp subsets of* X *and* Y*, respectively. Let* T *be a t-norm,* S *be a t-conorm. Then for the membership functions of fuzzy extensions of* R *it holds:*

(i) $\mu_{\Psi^T(R)}(A, B) = 1$ *if and only if there exist* $a \in A$ *and* $b \in B$ *such that* $\mu_R(a, b) = 1$*;*

(ii) $\mu_{\Psi_S(R)}(A, B) = 1$ *if and only if for every* $a \in A$ *and every* $b \in B$ *it holds* $\mu_R(a, b) = 1$*;*

(iii) $\mu_{\Psi^{T,S}(R)}(A, B) = 1$ *if and only if there exists* $a \in A$ *such that* $\mu_R(a, b) = 1$ *for every* $b \in B$*;*

(iv) $\mu_{\Psi_{T,S}(R)}(A, B) = 1$ *if and only if for every* $b \in B$ *there exists* $a \in A$ *such that* $\mu_R(a, b) = 1$*;*

(v) $\mu_{\Psi^{S,T}(R)}(A,B) = 1$ *if and only if there exists* $b \in B$ *such that* $\mu_R(a,b) = 1$ *for every* $a \in A$;

(vi) $\mu_{\Psi_{S,T}(R)}(A,B) = 1$ *if and only if for every* $a \in A$ *there exists* $b \in B$ *such that* $\mu_R(a,b) = 1$.

PROOF. (i) By (6.28), we obtain

$$\begin{aligned}\mu_{\Psi^T(R)}(A,B) &= \sup\{T(T(\mu_A(x),\mu_B(y)),\mu_R(x,y)) \mid x \in X,\ y \in Y\}\\ &= \sup\{T(T(\chi_A(x),\chi_B(y)),\mu_R(x,y)) \mid x \in X,\ y \in Y\}\\ &= \begin{cases} 1 & \text{if } \mu_R(a,b)=1, \text{ for some } a \in A \text{ and } b \in B,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

The other statements can be proved analogously. The detailed proofs can be found in [104]. ■

In the following proposition we prove some duality results between fuzzy extensions of valued relations (6.34). In a special case, particularly $T = \min$ and $S = \max$, the analogical results can be also found in [41].

PROPOSITION 6.29 *Let* X, Y *be nonempty sets,* T *be a t-norm,* S *be a t-conorm dual to* T. *Then*

(i) Ψ^T *is dual to* Ψ_S,

(ii) $\Psi^{T,S}$, *is dual to* $\Psi_{S,T}$,

(iii) $\Psi^{S,T}$ *is dual to* $\Psi_{T,S}$.

PROOF. (i) Let $R \in \mathcal{F}(X \times Y)$, we have to prove (6.24), i.e.,

$$\Psi^T(\mathcal{C}R) = \mathcal{C}\Psi_S(R). \tag{6.35}$$

To prove (6.35), let $A \in \mathcal{F}(X), B \in \mathcal{F}(Y)$. We have to show that

$$\mu_{\Psi^T(\mathcal{C}R)}(A,B) = \mu_{\mathcal{C}\Psi_S(R)}(A,B).$$

By definition (6.28) and by duality of T and S, see (4.7), (4.8), we obtain

$$\begin{aligned}&\mu_{\Psi^T(\mathcal{C}R)}(A,B)\\ &\quad= \sup\{T(T(\mu_A(x),\mu_B(y)),\mu_{\mathcal{C}R}(x,y)) \mid x \in X,\ y \in Y\}\\ &\quad= \sup\{1 - S(1 - T(\mu_A(x),\mu_B(y)),\mu_R(x,y)) \mid x \in X,\ y \in Y\}\\ &\quad= 1 - \inf\{S(S(\mu_{\mathcal{C}A}(x),\mu_{\mathcal{C}B}(y)),\mu_R(x,y)) \mid x \in X,\ y \in Y\}\\ &\quad= 1 - \mu_{\Psi_S(R)}(A,B) = \mu_{\mathcal{C}\Psi_S(R)}(A,B).\end{aligned}$$

Statements (ii) and (iii) can be proved analogously. ■

A number of other properties in case of $T = \min$ and $S = \max$ can be found in [41].

Some more properties of the fuzzy extensions of valued relations for the case $X = Y = \mathbf{R}^m$ shall be derived in the last section of this chapter.

8. Fuzzy Quantities and Fuzzy Numbers

In this section, we are concerned with fuzzy subsets of the real line. Therefore we have $X = \mathbf{R}$ and $\mathcal{F}(X) = \mathcal{F}(\mathbf{R})$.

DEFINITION 6.30

(i) *A fuzzy subset* $A = \{A_\alpha\}_{\alpha \in [0,1]}$ *of* $\mathbf{R}$ *is called a* fuzzy quantity. *The set of all fuzzy quantities will be denoted by* $\mathcal{F}(\mathbf{R})$.

(ii) *A fuzzy quantity* $A = \{A_\alpha\}_{\alpha \in [0,1]}$ *is called a* fuzzy interval *if* A_α *is nonempty and convex subset of* $\mathbf{R}$ *for all* $\alpha \in [0,1]$. *The set of all fuzzy intervals will be denoted by* $\mathcal{F}_I(\mathbf{R})$.

(iii) *A fuzzy interval A is called a* fuzzy number *if its core is a singleton. The set of all fuzzy numbers will be denoted by* $\mathcal{F}_N(\mathbf{R})$.

Notice that the membership function $\mu_A : \mathbf{R} \to [0,1]$ of a fuzzy interval A is quasiconcave on $\mathbf{R}$, that is, for all $x, y \in \mathbf{R}$, $x \neq y$, $\lambda \in (0,1)$, the following inequality holds:

$$\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}.$$

By Definition 6.30, each fuzzy interval is normalized, since $\mathrm{Core}(A) = [A]_1$ is nonempty, that is, there exists an element $x_0 \in \mathbf{R}$ with $\mu_A(x_0) = 1$. Then $\mathrm{Hgt}(A) = 1$. Moreover, the restriction of the membership function μ_A to $(-\infty, x_0]$ is non-decreasing and the restriction of μ_A to $[x_0, +\infty)$ is a non-increasing function.

Now we focus our attention to some subclasses of the class of fuzzy intervals that turn out to be useful in applications.

A *closed fuzzy interval* A has an upper semicontinuous membership function μ_A or, equivalently, for each $\alpha \in (0,1]$ the α-cut $[A]_\alpha$ is a closed subinterval in $\mathbf{R}$. Such a membership function μ_A, and the corresponding fuzzy interval A, can be fully described by a quadruple (l, r, F, G), where $l, r, \in \mathbf{R}$ with $l \leq r$, and F, G are non-increasing left continuous functions mapping $(0, +\infty)$ into $[0,1)$, by setting

$$\mu_A(x) = \begin{cases} F(l-x) & \text{if } x \in (-\infty, l), \\ 1 & \text{if } x \in [l, r], \\ G(x-r) & \text{if } x \in (r, +\infty). \end{cases} \tag{6.36}$$

We shall briefly write $A = (l, r, F, G)$ and the set of all closed fuzzy intervals will be denoted by $\mathcal{F}_{CI}(\mathbf{R})$. As the ranges of F and G are included in $[0,1)$, we have $\mathrm{Core}(A) = [l, r]$. We can see that the functions F, G describe the left and right "shape" of μ_A, respectively. Observe also that each crisp number $x_0 \in \mathbf{R}$ and each crisp interval $[a, b] \subset \mathbf{R}$ belongs to $\mathcal{F}_{CI}(\mathbf{R})$, as they may be

equivalently expressed by the characteristic functions $\chi_{\{x_0\}}$ and $\chi_{[a,b]}$, respectively. These characteristic functions can be also described in the form (6.36) with $F(x) = G(x) = 0$ for all $x \in (0, +\infty)$.

EXAMPLE 6.31 *(Gaussian fuzzy number)* Let $a \in \mathbf{R}$, $\gamma \in (0, +\infty)$, and let $A = (a, a, G, G)$ where

$$G(x) = e^{-\frac{x^2}{\gamma}}.$$

Then the membership function μ_A of A is given by

$$\mu_A(x) = G(x - a) = e^{-\frac{(x-a)^2}{\gamma}},$$

see Figure 6.1, where $\gamma = 2, a = 3$. □

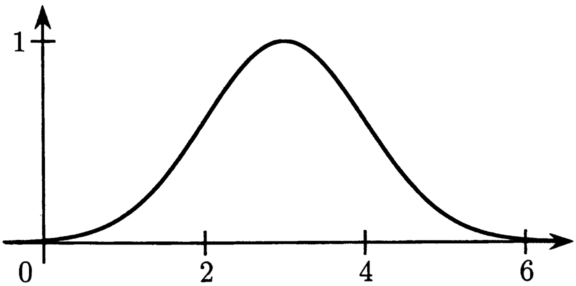

Figure 6.1.

A class of more specific fuzzy intervals of $\mathcal{F}_{CI}(\mathbf{R})$ is obtained, if the α-cuts are required to be bounded intervals. Let $l, r, \in \mathbf{R}$ with $l \leq r$, let $\gamma, \delta \in [0, +\infty)$ and let L, R be non-increasing non-constant functions mapping interval $(0, 1]$ into $[0, +\infty)$, i.e., $L, R : (0, 1] \to [0, +\infty)$. Moreover, assume that $L(1) = R(1) = 0$, define $L(0) = \lim_{x \to 0} L(x)$, $R(0) = \lim_{x \to 0} R(x)$, and for each $x \in \mathbf{R}$ let

$$\mu_A(x) = \begin{cases} L^{(-1)}\left(\frac{l-x}{\gamma}\right) & \text{if } x \in (l - \gamma, l), \gamma > 0, \\ 1 & \text{if } x \in [l, r], \\ R^{(-1)}\left(\frac{x-r}{\delta}\right) & \text{if } x \in (r, r + \delta), \delta > 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $L^{(-1)}$, $R^{(-1)}$ are pseudo-inverse functions of L, R, respectively. We shall write $A = (l, r, \gamma, \delta)_{LR}$, and say that A is an (L, R)*-fuzzy interval*. The set of all (L, R)-fuzzy intervals will be denoted by $\mathcal{F}_{LR}(\mathbf{R})$; see also [57]. The values of γ, δ are called the *left* and the *right spread of* A, respectively.

Observe that $\mathrm{Supp}(A) = [l-\gamma, r+\delta]$, $\mathrm{Core}(A) = [l, r]$ and $[A]_\alpha$ is a compact interval for every $\alpha \in (0, 1]$.

Particularly important fuzzy intervals are so called *trapezoidal fuzzy intervals* where $L(x) = R(x) = 1 - x$ for all $x \in [0, 1]$. In this case, the subscript LR will be omitted in the notation. If $l = r$, then $A = (r, r, \gamma, \delta)$ is called a *triangular fuzzy number* and the notation is simplified to: $A = (r, \gamma, \delta)$.

Interesting classes of fuzzy quantities are based on the concept of a basis of generators; see [61], [60].

DEFINITION 6.32 *A fuzzy quantity A given by the membership function $\mu_A : \mathbf{R} \to [0, 1]$ is called a* generator in $\mathbf{R}$ *if*

(i) $0 \in \mathrm{Core}(A)$,

(ii) μ_A *is quasiconcave on* $\mathbf{R}$.

Notice that each generator is a special fuzzy interval A that satisfies (i).

DEFINITION 6.33 *A set $\mathcal{B}$ of generators in $\mathbf{R}$ is called a* basis of generators in $\mathbf{R}$ *if*

(i) $\chi_{\{0\}} \in \mathcal{B}$, $\chi_{\mathbf{R}} \in \mathcal{B}$,

(ii) $\max\{f, g\} \in \mathcal{B}$ *and* $\min\{f, g\} \in \mathcal{B}$ *whenever* $f, g \in \mathcal{B}$.

DEFINITION 6.34 *Let $\mathcal{B}$ be a basis of generators. A fuzzy quantity A given by the membership function $\mu_A : \mathbf{R} \to [0, 1]$ is called a $\mathcal{B}$-*fuzzy interval *if there exists $a_A \in \mathbf{R}$ and $g_A \in \mathcal{B}$ such that for each $x \in \mathbf{R}$*

$$\mu_A(x) = g_A(x - a_A).$$

The set of all $\mathcal{B}$-fuzzy intervals will be denoted by $\mathcal{F}_{\mathcal{B}}(\mathbf{R})$. Each $A \in \mathcal{F}_{\mathcal{B}}(\mathbf{R})$ is represented by a pair (a_A, g_A), we write $A = (a_A, g_A)$. An ordering relation $\leq_{\mathcal{B}}$ is defined on $\mathcal{F}_{\mathcal{B}}(\mathbf{R})$ as follows: For $A, B \in \mathcal{F}_{\mathcal{B}}(\mathbf{R})$, $A = (a_A, g_A)$ and $B = (a_B, g_B)$, we write $A \leq_{\mathcal{B}} B$ if and only if

$$(a_A < a_B) \text{ or } (a_A = a_B \text{ and } g_A \leq g_B). \tag{6.37}$$

Notice that $\leq_{\mathcal{B}}$ is a partial ordering on $\mathcal{F}_{\mathcal{B}}(\mathbf{R})$. The proof of the following proposition follows directly from Definition 6.33.

PROPOSITION 6.35 *A pair $(\mathcal{B}, \leq)$, where $\mathcal{B}$ is a basis of generators and $\leq$ is the pointwise ordering of functions, is a lattice with the maximal element $\chi_{\mathbf{R}}$ and minimal element $\chi_{\{0\}}$.*

EXAMPLE 6.36 The following sets of functions form a basis of generators in $\mathbf{R}$:

(i) $\mathcal{B}_D = \{\chi_{\{0\}}, \chi_{\mathbf{R}}\}$ - discrete basis,

(ii) $\mathcal{B}_I = \{\chi_{[a,b]} \mid -\infty \leq a \leq b \leq +\infty\}$ - interval basis,

(iii) $\mathcal{B}_G = \{\mu \mid \mu(x) = g^{(-1)}(|x| / d)$ for each $x \in \mathbf{R}$, $d > 0\} \cup \{\chi_{\{0\}}, \chi_{\mathbf{R}}\}$, where $g : (0,1] \to [0,+\infty)$ is non-increasing non-constant function, $g(1) = 0$, $g(0) = \lim_{x\to 0} g(x)$. Evidently, the pointwise relation $\leq$ between function values is a linear ordering on $\mathcal{B}_G$.

□

EXAMPLE 6.37 $\mathcal{F}_{\mathcal{B}_G}(\mathbf{R}) = \{\mu \mid$ there exists $a \in \mathbf{R}$ and $g \in \mathcal{B}_G$, such that $\mu(x) = g(x-a)$ for each $x \in \mathbf{R}\}$. Evidently, the relation $\leq_{\mathcal{B}}$ is a linear ordering on $\mathcal{F}_{\mathcal{B}_G}(\mathbf{R})$. □

9. Fuzzy Extensions of Real-Valued Functions

Now, we shall deal with the problem of fuzzy extension of a real function f, where $f : \mathbf{R}^m \to \mathbf{R}$, $m \geq 1$, to a function $\tilde{f} : \mathcal{F}(\mathbf{R}) \times \cdots \times \mathcal{F}(\mathbf{R}) \to \mathcal{F}(\mathbf{R})$, applying the extension principle from Definition 6.11. Let $A_i \in \mathcal{F}(\mathbf{R})$ be fuzzy quantities given by the membership functions $\mu_{A_i} : \mathbf{R} \to [0,1]$, $i = 1, 2, \ldots, m$. Let T be a t-norm and let a fuzzy set $A \in \mathcal{F}(\mathbf{R}^m)$ be given by the membership function $\mu_A : \mathbf{R}^m \to [0,1]$, for all $x = (x_1, \ldots, x_m) \in \mathbf{R}^m$ as follows:

$$\mu_A(x) = T(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_m}(x_m)). \tag{6.38}$$

The fuzzy set $A \in \mathcal{F}(\mathbf{R}^m)$ given by the membership function (6.38) is called the *fuzzy vector of non-interactive fuzzy quantities*; see [47]. Applying (6.15), we obtain for all $y \in \mathbf{R}^m$:

$$\mu_{\tilde{f}(A)}(y) = \begin{cases} \sup\{\mu_A(x) \mid x \in \mathbf{R}^m, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{6.39}$$

Let $D = (d_1, d_2, \ldots, d_m)$ be a nonsingular $m \times m$ matrix, where all $d_i \in \mathbf{R}^m$ are column vectors, $i = 1, 2, \ldots, m$. Let a fuzzy set $B \in \mathcal{F}(\mathbf{R}^m)$ be given by the membership function $\mu_B : \mathbf{R}^m \to [0,1]$, for all $x = (x_1, \ldots, x_m) \in \mathbf{R}^m$ as follows:

$$\mu_B(x) = T(\mu_{A_1}(\langle d_1, x\rangle), \mu_{A_2}(\langle d_2, x\rangle), \ldots, \mu_{A_m}(\langle d_m, x\rangle)). \tag{6.40}$$

The fuzzy set $B \in \mathcal{F}(\mathbf{R}^m)$ given by the membership function (6.40) is called the *fuzzy vector of interactive fuzzy quantities*, or the *oblique fuzzy vector*, and the matrix D is called the *obliquity matrix*; see [47].

Notice that if D is equal to the identity matrix $E = (e_1, e_2, \ldots, e_m)$, $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is only at the ith position, then the corresponding vector of interactive fuzzy quantities is a noninteractive one. Interactive fuzzy numbers have been extensively studied, e.g., in [44], [47], [93] and [94]. In this book, we shall need them again in Chapter 9.

Now, we shall continue our investigation of the non-interactive fuzzy quantities.

EXAMPLE 6.38 Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined for all $(x_1, x_2) \in \mathbf{R}^2$ as follows: $f(x_1, x_2) = x_1 * x_2$, where $*$ is a binary operation on $\mathbf{R}$, e.g., one of the four arithmetic operations $(+, -, \cdot, /)$. Let $A_1, A_2 \in \mathcal{F}(\mathbf{R})$ be fuzzy quantities given by membership functions $\mu_{A_i} : \mathbf{R} \to [0, 1]$, $i = 1, 2$. Then, for a given t-norm T, the fuzzy extension $\tilde{f} : \mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}) \to \mathcal{F}(\mathbf{R})$ defined by (6.39) as

$$\mu_{A_1 \circledast_T A_2}(y) = \max\{0, \sup\{T(\mu_{A_1}(x_1), \mu_{A_2}(x_2)) \mid x_1 * x_2 = y\}\}$$

corresponds to the operation $\circledast_T$ on $\mathcal{F}(\mathbf{R})$. It is obvious that $\circledast_T$ is an extension of $*$, since for arbitrary crisp subsets $A_1, A_2 \in \mathcal{P}(\mathbf{R})$ we obtain

$$A_1 \circledast_T A_2 = A_1 * A_2, \tag{6.41}$$

and, as a special case thereof, for any two crisp numbers $a, b \in \mathbf{R}$,

$$a \circledast_T b = a * b.$$

If $A_1, A_2 \in \mathcal{F}(\mathbf{R})$ are fuzzy quantities, we obtain (6.41) in terms of α-cuts as follows

$$[A_1 \circledast_T A_2]_\alpha = [A_1]_\alpha * [A_2]_\alpha, \tag{6.42}$$

where $\alpha \in (0, 1]$, or in terms of mapping f, equality (6.42) can be written as

$$[\tilde{f}(A_1, A_2)]_\alpha = f([A_1]_\alpha, [A_2]_\alpha). \tag{6.43}$$

□

Now, we investigate equality (6.43) in a more general setting as a commutation of a diagram of two operations: mapping by f or $\tilde{f}$ and α-cutting of A or $\tilde{f}(A)$. Considering (6.38) and (6.39), we are interested in the following equality

$$[\tilde{f}(A)]_\alpha = f([A_1]_\alpha, \ldots, [A_m]_\alpha). \tag{6.44}$$

The process of forming of the left side and the right side of (6.44) may be visualized by the diagram depicted in Figure 6.2. Observe that, by (6.38) and (6.13), we obtain

$$A = A_1 \times_T A_2 \times_T \cdots \times_T A_m.$$

$$\begin{tikzcd}
{[\tilde{f}(A)]_\alpha} \arrow[r, phantom, "="] & f([A_1]_\alpha,\ldots,[A_m]_\alpha) \\
\tilde{f}(A) \arrow[u, "\alpha\text{-cutting}"] & {[A_1]_\alpha,\ldots,[A_m]_\alpha} \arrow[u, "\text{mapping}"] \\
A \arrow[u, "\text{mapping}"] \arrow[r, leftrightarrow] & A_1,\ldots,A_m \arrow[u, "\alpha\text{-cutting}"]
\end{tikzcd}$$

Figure 6.2.

If the equality at the top of this diagram is satisfied, we say, that the *diagram commutes*. For the beginning, we derive several results concerning some convexity properties of the individual elements in the diagram. The first result is a generalization of Proposition 4.45 for more than two membership functions. Notice that the membership functions in question are not assumed to be normalized.

PROPOSITION 6.39 *Let $A_i \in \mathcal{F}(\mathbf{R})$ be fuzzy quantities given by the membership functions $\mu_{A_i} : \mathbf{R} \to [0,1]$, $i = 1,2,\ldots,m$. Let T be a t-norm and let a fuzzy quantity $A \in \mathcal{F}(\mathbf{R}^m)$ be given by the membership function $\mu_A : \mathbf{R}^m \to [0,1]$ defined by (6.38). If μ_{A_i} are T-quasiconcave on $\mathbf{R}$ for all $i = 1,2,\ldots,m$, then μ_A is T-quasiconcave on $\mathbf{R}^m$.*

PROOF. The proof can be analogous to the proof of Corollary 4.46 for $m = 2$. ■

If we assume that $A_i \in \mathcal{F}(\mathbf{R})$ are normalized, then by Propositions 4.28 and 4.29, T-quasiconcavity on $\mathbf{R}$ is equivalent to quasiconcavity of μ_A on $\mathbf{R}$. Another proposition follows from Proposition 5.25.

PROPOSITION 6.40 *Let $A_i \in \mathcal{F}_I(\mathbf{R})$ be fuzzy intervals given by the membership functions $\mu_{A_i} : \mathbf{R} \to [0,1]$, $i = 1,2,\ldots,m$. Let $G = \{G_k\}_{k=1}^{\infty}$ be an aggregation operator and let $A \in \mathcal{F}(\mathbf{R}^m)$ be given by the membership function $\mu_A : \mathbf{R}^m \to [0,1]$ for all $x \in \mathbf{R}^m$ by*

$$\mu_A(x) = G_m(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_m}(x_m)).$$

Then μ_A is upper-starshaped on $\mathbf{R}^m$.

PROOF. Let us define for $i = 1, 2, \ldots, m$

$$\mu_i(x_1, \ldots, x_m) = \mu_{A_i}(x_i). \tag{6.45}$$

Then μ_i is normalized and quasiconcave on $\mathbf{R}^m$. In order to apply the proof of Proposition 5.25 we have to show that $\text{Core}(\mu_1) \cap \cdots \cap \text{Core}(\mu_m) \neq \emptyset$. We prove even more, particularly

$$\text{Core}(\mu_1) \cap \cdots \cap \text{Core}(\mu_m) = \text{Core}(\mu_{A_1}) \times \cdots \times \text{Core}(\mu_{A_m}) \neq \emptyset.$$

Indeed, if $x = (x_1, \ldots, x_m) \in \text{Core}(\mu_1) \cap \cdots \cap \text{Core}(\mu_m)$, then for all $i = 1, 2, \ldots, m$, $\mu_i(x) = 1$ and by (6.45) we obtain $\mu_{A_i}(x_i) = 1$. Consequently, for all $i = 1, 2, \ldots, m$, $x_i \in \text{Core}(\mu_{A_i})$; therefore, $x = (x_1, \ldots, x_m) \in \text{Core}(\mu_{A_1}) \times \cdots \times \text{Core}(\mu_{A_m})$.

Conversely, let $x = (x_1, \ldots, x_m) \in \text{Core}(\mu_{A_1}) \times \cdots \times \text{Core}(\mu_{A_m})$. Then for each $x_i \in \text{Core}(\mu_{A_i})$ and all $i = 1, 2, \ldots, m$, and by (6.45) it follows that $\mu_i(x) = 1$ for all i. Thus we obtain

$$x = (x_1, \ldots, x_m) \in \text{Core}(\mu_1) \cap \cdots \cap \text{Core}(\mu_m).$$

Finally, since by the assumption we have $A_i \in \mathcal{F}_I(\mathbf{R})$, and $\text{Core}(\mu_{A_i})$ is nonempty for all $i = 1, 2, \ldots, m$, we have

$$\text{Core}(\mu_{A_1}) \times \cdots \times \text{Core}(\mu_{A_m}) \neq \emptyset.$$

The rest of the proposition can be proved analogously to Proposition 5.25 with G being an aggregation operator. Thus, $\mu_A = G_m(\mu_{A_1}, \ldots, \mu_{A_m})$ is upper-starshaped on $\mathbf{R}^m$. ■

The next example shows that Proposition 6.42 cannot be strengthen in such a way that μ_A is quasiconcave on $\mathbf{R}^m$.

EXAMPLE 6.41 Let $X = \mathbf{R}^2$ and let $\mu_{A_i} : \mathbf{R} \to [0, 1], i = 1, 2$, be defined as follows:

$$\mu_{A_1}(x_1) = \max\left\{0, 1 - \sqrt{|x_1|}\right\}, \ \mu_{A_2}(x_2) = \max\left\{0, 1 - \sqrt{|x_2|}\right\}.$$

Let $T = T_P$ be the product t-norm. Following (6.38) define for all $(x_1, x_2) \in X$:

$$\mu_A(x_1, x_2) = \max\left\{0, 1 - \sqrt{|x_1|}\right\} \cdot \max\left\{0, 1 - \sqrt{|x_2|}\right\}. \tag{6.46}$$

It is evident that μ_{A_i} is normalized quasiconcave functions on $\mathbf{R}$ for $i = 1, 2$. By Proposition (6.40), μ_A defined by (6.46) is upper starshaped. In Figure 6.3, the contours of some α-cuts of the fuzzy set A given by (6.46), are depicted.

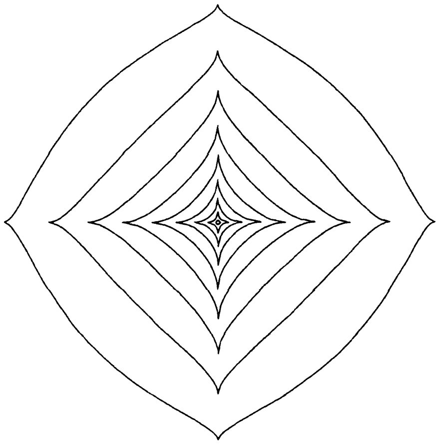

Figure 6.3.

This figure demonstrates that μ_A is not quasiconcave on X, as some of its α-cuts are not convex. This fact can be verified by looking closely at the curves $\mu_A(x_1, x_2) = \alpha$ for $\alpha \in (0, 1]$. All α-cuts are, however, starshaped sets. □

The next two results concern the α-cuts of the fuzzy quantities.

PROPOSITION 6.42 *Let $A_i \in \mathcal{F}(\mathbf{R})$ be fuzzy quantities given by the membership functions $\mu_{A_i} : \mathbf{R} \to [0, 1]$, $i = 1, 2, \ldots, m$. Let T be a t-norm and let a fuzzy quantity $A \in \mathcal{F}(\mathbf{R}^m)$ be given by the membership function $\mu_A : \mathbf{R}^m \to [0, 1]$ in (6.38).*

(i) *If $\alpha \in (0, 1]$, then*

$$[A]_\alpha \subset [A_1]_\alpha \times [A_2]_\alpha \times \cdots \times [A_m]_\alpha. \tag{6.47}$$

(ii) *The equality*

$$[A]_\alpha = [A_1]_\alpha \times [A_2]_\alpha \times \cdots \times [A_m]_\alpha, \tag{6.48}$$

holds for all $\alpha \in (0, 1]$, if and only if $T = T_M$.

PROOF. (i) Let $x = (x_1, \ldots, x_m) \in [A]_\alpha$, i.e.,

$$\mu_A(x) = T(\mu_{A_1}(x_1), \ldots, \mu_{A_m}(x_m)) \geq \alpha.$$

Since by (4.6)

$$\min\{\mu_{A_1}(x_1), \ldots, \mu_{A_m}(x_m)\} \geq T(\mu_{A_1}(x_1), \ldots, \mu_{A_m}(x_m)),$$

we obtain $\mu_{A_i}(x_i) \geq \alpha$ for all $i = 1, 2, \ldots, m$. Consequently, for all $i = 1, 2, \ldots, m$, we have $x_i \in [A_i]_\alpha$ and also $x = (x_1, \ldots, x_m) \in [A_1]_\alpha \times [A_2]_\alpha \times \cdots \times [A_m]_\alpha$.

(ii) Let $T \neq \min$. Then there exists $x = (x_1, \ldots, x_m) \in \mathbf{R}^m$ such that $\min\{\mu_{A_1}(x_1), \ldots, \mu_{A_m}(x_m)\} > T(\mu_{A_1}(x_1), \ldots, \mu_{A_m}(x_m))$.

Putting $\beta = \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_m}(x_m)\}$, we have $\beta > 0$ and $x_i \in [A_i]_\beta$ for all $i = 1, 2, \ldots, m$, i.e., $x = (x_1, \ldots, x_m) \in [A_1]_\beta \times [A_2]_\beta \times \cdots \times [A_m]_\beta$. However, $\mu_A(x) = T(\mu_{A_1}(x_1), \ldots, \mu_{A_m}(x_m)) < \beta$ and therefore $x = (x_1, \ldots, x_m) \notin [A]_\beta$, a contradiction with (6.48). Thus, $T = \min$.

On the other hand, if $T = \min$, then

$$\mu_A = \min\{\mu_{A_1}, \ldots, \mu_{A_m}\}. \tag{6.49}$$

Let $\alpha \in (0, 1]$ be arbitrary and $x = (x_1, \ldots, x_m) \in \mathbf{R}^m$ be also arbitrary with $x \in [A_1]_\alpha \times [A_2]_\alpha \times \cdots \times [A_m]_\alpha$. Then $\mu_{A_i}(x_i) \geq \alpha$ for all $i = 1, 2, \ldots, m$ and it follows that $\min\{\mu_{A_1}(x_1), \ldots, \mu_{A_m}(x_m)\} \geq \alpha$. Hence, by (6.49) we have $x \in [A]_\alpha$. We have just proved inclusion $[A]_\alpha \supset [A_1]_\alpha \times [A_2]_\alpha \times \cdots \times [A_m]_\alpha$, the opposite inclusion (6.47) is true by (i). Consequently, we have the required result (6.48). ■

Now, we shall deal with a fuzzy extension of a mapping f by using the extension principle (6.39). Some sufficient conditions under which $\tilde{f}(A)$ is quasiconcave on $\mathbf{R}$ will be given in the next section as the consequence of a more general result. The problem of commuting of the diagram in Figure 6.2 will be also resolved.

10. Higher Dimensional Fuzzy Quantities

In the previous section we assumed that the fuzzy subset $A \in \mathcal{F}(\mathbf{R}^m)$ was given in the special form (6.38), or eventually (6.40). In this section, we shall investigate fuzzy subsets of the m-dimensional real vector space $\mathbf{R}^m$, where m is a positive integer. The set of all fuzzy subsets of $\mathbf{R}^m$, denoted by $\mathcal{F}(\mathbf{R}^m)$, is called the set of *$m$-dimensional fuzzy quantities*. Sometimes the expression m-dimensional is omitted. We shall investigate the problem of extension a function $f : \mathbf{R}^m \to \mathbf{R}$ to a function $\tilde{f} : \mathcal{F}(\mathbf{R}^m) \to \mathcal{F}(\mathbf{R})$. The process of commuting of the operations of mapping and α-cutting is depicted on the diagram in Figure 6.4.

The following definition will be useful.

DEFINITION 6.43 *A fuzzy subset $A = \{A_\alpha\}_{\alpha \in [0,1]}$ of $\mathbf{R}^m$ is called* closed, bounded, compact *or* convex *if A_α is a closed, bounded, compact or convex subset of $\mathbf{R}^m$ for every $\alpha \in (0, 1]$, respectively.*

$$
\begin{tikzcd}
{[\tilde{f}(A)]_\alpha} \arrow[rr, phantom, "="] & & f([A]_\alpha) \\
\tilde{f}(A) \arrow[u, "\alpha\text{-cutting}"] & & {[A]_\alpha} \arrow[u, "\text{mapping}"] \\
& A \arrow[ul, "\text{mapping}"] \arrow[ur, "\alpha\text{-cutting}"'] &
\end{tikzcd}
$$

Figure 6.4.

If a fuzzy subset A of $\mathbf{R}^m$ given by the membership function $\mu_A : \mathbf{R}^m \to [0, 1]$ is closed, bounded, compact or convex, then $[A]_\alpha$ is a closed, bounded, compact or convex subset of $\mathbf{R}^m$ for every $\alpha \in (0, 1]$, respectively. Notice that A is convex if and only if its membership function μ_A is quasiconcave on $\mathbf{R}^m$.

In what follows we shall use the following important condition requiring that special optimization problems always posses some optimal solution. Some sufficient conditions securing this requirement will be presented later.

DEFINITION 6.44 Condition (C): *Let* $f : \mathbf{R}^m \to \mathbf{R}$, $\mu : \mathbf{R}^m \to [0, 1]$. *We say that* condition (C) is satisfied for f and μ, *if for every* $y \in \mathrm{Ran}(f)$ *there exists* $x_y \in \mathbf{R}^m$ *such that* $f(x_y) = y$ *and*

$$\mu(x_y) = \sup\{\mu(x) \mid x \in \mathbf{R}^m, f(x) = y\}. \tag{6.50}$$

THEOREM 6.45 *Let* $A \in \mathcal{F}(\mathbf{R}^m)$ *be a fuzzy quantity, let* μ_A *be upper-quasiconnected on* $\mathbf{R}^m$, *let* $f : \mathbf{R}^m \to \mathbf{R}$ *be continuous on* $\mathbf{R}^m$ *and let condition (C) be satisfied for* f *and* μ_A. *Then the membership function of* $\tilde{f}(A)$ *is quasiconcave on* $\mathbf{R}$.

PROOF. Let $\alpha \in (0, 1]$. We show that $[\tilde{f}(A)]_\alpha$ is convex. Let $y_i \in [\tilde{f}(A)]_\alpha$, $i = 1, 2$, with $y_1 < y_2$ and $\lambda \in (0, 1)$. Putting $y_0 = \lambda y_1 + (1 - \lambda)y_2$, we have $y_1 < y_0 < y_2$. (If $y_1 = y_2$, then there is nothing to prove.)

By Condition (C) there exists $x_i \in \mathbf{R}^m$, $i = 1, 2$, with $f(x_i) = y_i$ such that by (6.50) and (6.17) we get $\mu_A(x_i) = \mu_{\tilde{f}(A)}(y_i) \geq \alpha$, therefore, $x_i \in [A]_\alpha$.

Since μ_A is upper quasiconnected on $\mathbf{R}^m$, $[A]_\alpha$ is path-connected, therefore there exists a path P belonging to $[A]_\alpha$, i.e.,

$$P \subset [A]_\alpha. \tag{6.51}$$

Since P is connected and f is continuous on P with $f(x_i) = y_i$ and $y_1 < y_0 < y_2$, then $f(P)$ is also connected, $y_1, y_0, y_2 \in f(P)$ and it follows that there exists $x_0 \in P$ such that $f(x_0) = y_0$. By (6.51) we have $x_0 \in [A]_\alpha$, i.e., $\mu_A(x_0) \geq \alpha$, which implies $\mu_{\tilde{f}(A)}(y_0) = \sup\{\mu_A(x) \mid x \in \mathbf{R}^m, f(x) = y_0\} \geq \mu_A(x_0) \geq \alpha$. Consequently, $y_0 \in [\tilde{f}(A)]_\alpha$, thus $[\tilde{f}(A)]_\alpha$ is convex. ∎

Now, we return back to the question concerning sufficient conditions for the validity of Condition (C).

PROPOSITION 6.46 *Let $A \in \mathcal{F}(\mathbf{R}^m)$ be a compact fuzzy quantity and let $f : \mathbf{R}^m \to \mathbf{R}$ be a continuous function. Then Condition (C) is satisfied for f and μ_A.*

PROOF. Let $y \in \mathrm{Ran}(f)$ and denote $X_y = \{x \in \mathbf{R}^m \mid f(x) = y\}$. Then X_y is nonempty and closed. Put

$$\alpha = \sup\{\mu_A(x) \mid x \in X_y\}. \tag{6.52}$$

Without loss of generality we assume that $\alpha > 0$. Take a number β, $0 < \beta < \alpha$ such that $\alpha - \frac{\beta}{k} > 0$ for all $k = 1, 2, \ldots$, and denote

$$U_k = \left\{ x \in \mathbf{R}^m \;\middle|\; \mu_A(x) \geq \alpha - \frac{\beta}{k} \right\}, \quad k = 1, 2, \ldots. \tag{6.53}$$

By the compactness of $[A]_\delta$ for all $\delta \in (0, 1]$ we know that all U_k are compact and $U_{k+1} \subset U_k$ for all $k = 1, 2, \ldots$. Putting $V_k = U_k \cap X_y$ we obtain by (6.52) and (6.53) that V_k is nonempty, compact and $V_{k+1} \subset V_k$ for all $k = 1, 2, \ldots$. From the well known property of compact spaces it follows that $\bigcap_{k=1}^{\infty} V_k$ is nonempty. Hence, for any $x_y \in \bigcap_{k=1}^{\infty} V_k$ it holds: $f(x_y) = y$ and $\mu_A(x_y) \geq \alpha$. ∎

Clearly, the fuzzy set A is compact, if the α-cuts $[A]_\alpha$ are compact for all $\alpha \in (0, 1]$, or the α-cuts $[A]_\alpha$ are bounded for all $\alpha \in (0, 1]$ and the membership function μ_A is upper semicontinuous on $\mathbf{R}^m$.

Returning back to the problem formulated at the end of the last section, namely, the problem of the existence of sufficient conditions under which the membership function of $\tilde{f}(A)$ is quasiconcave on $\mathbf{R}$ with μ_A defined by (6.38), we have the following result.

THEOREM 6.47 *Let $A_i \in \mathcal{F}_I(\mathbf{R})$ be compact fuzzy intervals. Let T be a continuous t-norm and let a fuzzy quantity $A \in \mathcal{F}(\mathbf{R}^m)$ be given by the membership function $\mu_A : \mathbf{R}^m \to [0,1]$ as*

$$\mu_A(x) = T(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_m}(x_m)),$$

for all $x = (x_1, \ldots, x_m) \in \mathbf{R}^m$. Moreover, let $f : \mathbf{R}^m \to \mathbf{R}$ be continuous on $\mathbf{R}^m$. Then the membership function of $\tilde{f}(A)$ given by (6.39) is quasiconcave on $\mathbf{R}$.

PROOF. It is sufficient to show that $\mu_A(x)$ is upper-quasiconnected and $[A]_\alpha$ are compact for all $\alpha \in (0,1]$. Having this, the result follows from Proposition 6.46 and Theorem 6.45.

By Proposition 6.40, μ_A is upper starshaped on $\mathbf{R}^m$, hence μ_A is upper connected. As T is continuous, $[A]_\alpha$ is closed for all $\alpha \in (0,1]$.

It is also supposed that $[A_i]_\alpha$ are compact for all $\alpha \in (0,1], i = 1, 2, \ldots, m$, therefore the same holds for a Cartesian product $[A_1]_\alpha \times [A_2]_\alpha \times \cdots \times [A_m]_\alpha$. Applying (6.47), we obtain that all $[A]_\alpha$ are bounded, hence compact.

Finally, all assumptions of Proposition 6.46 are satisfied, thus Condition (C) is satisfied and by applying Theorem 6.45 we obtain the required result. ■

Now, we resolve the former problem of commuting of the diagrams in Figure 6.2 and Figure 6.4.

PROPOSITION 6.48 *Let $A \in \mathcal{F}(\mathbf{R}^m)$ be a fuzzy quantity, let μ_A be upper-quasiconnected on $\mathbf{R}^m$, let $f : \mathbf{R}^m \to \mathbf{R}$ be continuous on $\mathbf{R}^m$. Then $f([A]_\alpha)$ is convex for each $\alpha \in [0,1]$.*

PROOF. Let $\alpha \in [0,1]$. As $[A]_\alpha$ is path-connected and f is continuous, we conclude that $f([A]_\alpha)$ is a connected subset of $\mathbf{R}$, thus it is a convex subset of $\mathbf{R}$. ■

PROPOSITION 6.49 *Let $A \in \mathcal{F}(\mathbf{R}^m)$ be a fuzzy set, let $f : \mathbf{R}^m \to \mathbf{R}$. Then*

$$f([A]_\alpha) \subset [\tilde{f}(A)]_\alpha,$$

for each $\alpha \in (0,1]$.

PROOF. Let $\alpha \in (0,1]$ and $y \in f([A]_\alpha)$. Then there exists $x_y \in [A]_\alpha$, such that $f(x_y) = y$. By (6.39) we obtain

$$\mu_{\tilde{f}(A)}(y) = \sup\{\mu_A(x) \mid x \in \mathbf{R}^m, f(x) = y\} \geq \mu_A(x_y) \geq \alpha.$$

Hence, $y \in [\tilde{f}(A)]_\alpha$. ■

The following theorem gives a necessary and sufficient condition for the diagram in Figure 6.4 to commute.

THEOREM 6.50 *Let $A \in \mathcal{F}(\mathbf{R}^m)$ be a fuzzy quantity. Condition (C) is satisfied if and only if*

$$[\tilde{f}(A)]_\alpha = f([A]_\alpha), \tag{6.54}$$

for each $\alpha \in (0,1]$.

PROOF. 1. Let Condition (C) be satisfied. We have to prove only $[\tilde{f}(A)]_\alpha \subset f([A]_\alpha)$, the opposite inclusion holds by Proposition 6.49.

Let $\alpha \in (0,1]$ and $y \in [\tilde{f}(A)]_\alpha$. Then $\mu_{\tilde{f}(A)}(y) \geq \alpha$ and by Condition (C) we have

$$\mu_{\tilde{f}(A)}(y) = \sup\{\mu_A(x) \mid x \in \mathbf{R}^m, f(x) = y\} = \mu_A(x_y)$$

for some $x_y \in \mathbf{R}^m$ with $f(x_y) = y$. Combining these results we obtain $x_y \in [A]_\alpha$; consequently, $f(x_y) = y \in f([A]_\alpha)$.

2. On the contrary, suppose that Condition (C) does not hold. Then there exists y_0 such that for each $z \in \mathbf{R}^m$ with $f(z) = y_0$ we have

$$\sup\{\mu_A(x) \mid x \in \mathbf{R}^m, f(x) = y_0\} > \mu_A(z). \tag{6.55}$$

Put $\beta = \sup\{\mu_A(x) \mid x \in \mathbf{R}^m, f(x) = y_0\}$. Then $\mu_{\tilde{f}(A)}(y_0) = \beta$, i.e., $y_0 \in [\tilde{f}(A)]_\beta$. Suppose that (6.54) holds for $\alpha = \beta$. It follows that there exists $x_0 \in [A]_\beta$, i.e., $\mu_A(x_0) \geq \beta$, with $f(x_0) = y_0$. However, this is in contradiction with (6.55). ■

Theorem 6.50 is a reformulation of the well known Nguyen's result, see [76]. As a consequence of the Theorems 6.50, 6.47 and Proposition 6.46, we resolve the problem of commuting of the diagram in Figure 6.2.

THEOREM 6.51 *Let $A_i \in \mathcal{F}_I(\mathbf{R})$ be compact fuzzy intervals, $i = 1, 2, ..., m$. Let $f : \mathbf{R}^m \to \mathbf{R}$ be a continuous function, let T be a continuous t-norm and let a fuzzy quantity $A \in \mathcal{F}(\mathbf{R}^m)$ be given by the membership function $\mu_A : \mathbf{R}^m \to [0,1]$ by (6.38). Then*

$$[\tilde{f}(A)]_\alpha = f([A_1]_\alpha, \ldots, [A_m]_\alpha),$$

for each $\alpha \in (0,1]$.

PROOF. It is sufficient to show that

$$\mu_A(x) = T(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_m}(x_m))$$

is upper starshaped on $\mathbf{R}^m$ with $[A]_\alpha$ compact for all $\alpha \in (0,1]$. Then the rest of the proof follows from Proposition 6.46 and Theorem 6.50.

Indeed, by Proposition 6.40, μ_A is upper starshaped on $\mathbf{R}^m$. As μ_{A_i} are upper semicontinuous, $i = 1, 2, \ldots, m$, and T is continuous, it follows that $\mu_A = T(\mu_{A_1}, \ldots, \mu_{A_m})$ is upper semicontinuous on $\mathbf{R}^m$. Hence, $[A]_\alpha$ is closed for each $\alpha \in (0, 1]$.

It is supposed that $[A_i]_\alpha$ is compact for all $\alpha \in (0, 1]$, $i = 1, 2, \ldots, m$; the same holds for the Cartesian product $[A_1]_\alpha \times [A_2]_\alpha \times \cdots \times [A_m]_\alpha$ and applying (6.47), we obtain that $[A]_\alpha$ is bounded, thus compact.

Now, all assumptions of Proposition 6.46 are satisfied, thus Condition (C) holds and by Theorem 6.50 we obtain the required result. ∎

PROPOSITION 6.52 *Let $A \in \mathcal{F}(\mathbf{R}^m)$ be a compact fuzzy quantity. If $f : \mathbf{R}^m \to \mathbf{R}$ is continuous then $\tilde{f}(A)$ is compact.*

PROOF. Let $\alpha \in (0, 1]$. Since $[A]_\alpha$ is compact, then by continuity of f, it follows that $f([A]_\alpha)$ is compact. By Proposition 6.46, Condition (C) is satisfied and by Theorem 6.50 we obtain

$$[\tilde{f}(A)]_\alpha = f([A]_\alpha), \tag{6.56}$$

for each $\alpha \in (0, 1]$. ∎

COROLLARY 6.53 *If in Proposition 6.52, μ_A is upper-quasiconnected, then $[\tilde{f}(A)]_\alpha$ is a compact interval for each $\alpha \in (0, 1]$.*

PROOF. By Proposition 6.52, $[\tilde{f}(A)]_\alpha$ are compact and by Proposition 6.48, $f([A]_\alpha)$ are convex for all $\alpha \in (0, 1]$. Using equation (6.56), $[\tilde{f}(A)]_\alpha$ are convex and compact, i.e., compact intervals in $\mathbf{R}$. ∎

Commuting of the diagram in Figure 6.4 is important when calculating the extensions of aggregation operators in multi-criteria decision making, see Chapter 7.

11. Fuzzy Extensions of Valued Relations

In Section 6.6, we have introduced and investigated six types of fuzzy extensions of valued relations on $X \times Y$. In this section we shall deal with fuzzy extensions of valued and, particularly, binary relations on $\mathbf{R}^m$, where m is a positive integer, i.e., $X = Y = \mathbf{R}^m$. Binary relations can be viewed as special valued relations with values from $\{0, 1\}$. The usual component-wise equality relation $=$ and inequality relations $\leq$, $\geq$, $<$ and $>$ on $\mathbf{R}^m$ are simple examples of binary relations. The results derived in this section will be useful in fuzzy mathematical programming we shall investigate in Part II.

We start with three important examples.

EXAMPLE 6.54 Consider the usual binary relation = ("equal") on $\mathbf{R}^m$, given by the membership function $\mu_=$ for all $x, y \in \mathbf{R}^m$ as

$$\mu_=(x,y) = \begin{cases} 1 & \text{if } x_i = y_i \text{ for all } i = 1,2,\ldots,m, \\ 0 & \text{otherwise.} \end{cases}$$

Let T be a t-norm, A, B be fuzzy subsets of $\mathbf{R}^m$ with the corresponding membership functions $\mu : \mathbf{R}^m \to [0,1]$, $\nu : \mathbf{R}^m \to [0,1]$, respectively. Then by (6.25), the membership function $\mu_{\Psi^T(=)}$ of the T-fuzzy extension of relation = can be derived as

$$\begin{aligned} \mu_{\Psi^T(=)}(A,B) &= \sup\{T(\mu_=(x,y), T(\mu(x),\nu(y))) \mid x,y \in \mathbf{R}^m\} \\ &= \sup\{T(1, T(\mu(x),\nu(y))) \mid x,y \in \mathbf{R}^m,\ x = y\} \\ &= \sup\{T(\mu(x),\nu(x)) \mid x \in \mathbf{R}^m\} \\ &= \mathrm{Hgt}(A \cap_T B). \end{aligned}$$

□

EXAMPLE 6.55 Consider the usual binary relation $\geq$ ("greater than or equal to") on $\mathbf{R}^m$. The corresponding membership function is defined as

$$\mu_\geq(x,y) = \begin{cases} 1 & \text{if } x_i \geq y_i \text{ for all } i = 1,2,\ldots,m, \\ 0 & \text{otherwise.} \end{cases}$$

Let T be a t-norm, A, B be fuzzy subsets of $\mathbf{R}^m$ with the corresponding membership functions $\mu : \mathbf{R}^m \to [0,1]$, $\nu : \mathbf{R}^m \to [0,1]$, respectively. Then by (6.25), the membership function $\mu_{\Psi^T(\geq)}$ of the T-fuzzy extension of relation $\geq$ can be derived as follows:

$$\begin{aligned} \mu_{\Psi^T(\geq)}(A,B) &= \sup\{T(\mu_\geq(x,y), T(\mu(x),\nu(y))) \mid x,y \in \mathbf{R}^m\} \\ &= \sup\{T(1, T(\mu(x),\nu(y))) \mid x,y \in \mathbf{R}^m,\ x \geq y\} \\ &= \sup\{T(\mu(x),\nu(y)) \mid x,y \in \mathbf{R}^m,\ x \geq y\}. \end{aligned}$$

□

EXAMPLE 6.56 Let $d > 0$ and let $\varphi_d : \mathbf{R} \to [0,1]$ be a function defined as follows

$$\varphi_d(t) = \begin{cases} 1 & \text{if } t \geq 0, \\ \dfrac{d+t}{d} & \text{if } -d \leq t < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then the valued relation R_d defined by the membership function μ_{R_d} for all $x, y \in \mathbf{R}$ as

$$\mu_{R_d}(x,y) = \varphi_d(x-y)$$

is a "generalized" inequality relation $\geq$ on $\mathbf{R}$. By (6.25), the membership function $\mu_{\Psi^T(R_d)}$ of the T-fuzzy extension of relation R_d is given by

$$\mu_{\Psi^T(R_d)}(A,B) = \sup\{T(\varphi_d(x-y), T(\mu(x),\nu(y))) \mid x, y \in \mathbf{R}\}.$$

□

Now, we deal with properties of fuzzy extensions of binary relations on $\mathbf{R}^m$. We start with investigation of m-dimensional intervals. We consider the usual componentwise binary relations in $\mathbf{R}^m$, namely "less than or equal to" and "equal to", i.e., $R \in \{\leq, =\}$. For brevity, we simplify the notation for respective fuzzy extension of relations $\leq$ and $=$ as follows:

T-fuzzy extension	*Simplification*
$\Psi^T(\leq)$	$\tilde{\leq}^T$
$\Psi_S(\leq)$	$\tilde{\leq}_S$
$\Psi^{T,S}(\leq)$	$\tilde{\leq}^{T,S}$
$\Psi_{T,S}(\leq)$	$\tilde{\leq}_{T,S}$
$\Psi^{S,T}(\leq)$	$\tilde{\leq}^{S,T}$
$\Psi_{S,T}(\leq)$	$\tilde{\leq}_{S,T}$

T-fuzzy extension	*Simplification*
$\Psi^T(=)$,	$\tilde{=}^T$
$\Psi_S(=)$	$\tilde{=}_S$
$\Psi^{T,S}(=)$	$\tilde{=}^{T,S}$
$\Psi_{T,S}(=)$	$\tilde{=}_{T,S}$
$\Psi^{S,T}(=)$	$\tilde{=}^{S,T}$
$\Psi_{S,T}(=)$	$\tilde{=}_{S,T}$

THEOREM 6.57 *Let A, B be nonempty and closed intervals of $\mathbf{R}^m$, $A = \{a \in \mathbf{R}^m \mid \underline{a} \leq a \leq \overline{a}\}$, $B = \{b \in \mathbf{R}^m \mid \underline{b} \leq b \leq \overline{b}\}$. Let T be a t-norm, S be a t-conorm. Let $\leq$ and $=$ be usual binary relations in $\mathbf{R}^m$. Then*

(i) *1* $\mu_{\tilde{\leq}^T}(A,B) = 1$ *if and only if* $\underline{a} \leq \overline{b}$,

2 $\mu_{\tilde{=}^T}(A,B) = 1$ *if and only if* $\underline{a} \leq \overline{b}$ *and* $\overline{a} \geq \underline{b}$;

(ii) *1* $\mu_{\tilde{\leq}_S}(A,B) = 1$ *if and only if* $\overline{a} \leq \underline{b}$,

2 $\mu_{\tilde{=}_S}(A,B) = 1$ *if and only if* $\underline{a} = \underline{b} = \overline{b} = \overline{a}$;

(iii) *1* $\mu_{\tilde{\leq}^{T,S}}(A,B) = 1$ *if and only if* $\underline{a} \leq \underline{b}$,

2 $\mu_{\tilde{=}^{T,S}}(A,B) = 1$ *if and only if* $\underline{a} \leq \underline{b} = \overline{b} \leq \overline{a}$;

(iv) *1* $\mu_{\tilde{\leq}_{T,S}}(A,B) = 1$ *if and only if* $\underline{a} \leq \underline{b}$,

2 $\mu_{\tilde{=}_{T,S}}(A,B) = 1$ *if and only if* $\underline{a} \leq \underline{b} \leq \overline{b} \leq \overline{a}$;

(v) *1* $\mu_{\tilde{\leq}^{S,T}}(A,B) = 1$ *if and only if* $\overline{a} \leq \overline{b}$, $\mu_{\tilde{=}^{S,T}}(A,B) = 1$ *if and only if* $\underline{b} \leq \underline{a} = \overline{a} \leq \overline{b}$;

(vi) *1* $\mu_{\tilde{\leq}_{S,T}}(A,B) = 1$ *if and only if* $\overline{a} \leq \overline{b}$,

2 $\mu_{\tilde{=}_{S,T}}(A,B) = 1$ *if and only if* $\underline{b} \leq \underline{a} \leq \overline{a} \leq \overline{b}$.

PROOF.

(i) 1. Let $\mu_{\tilde{\leq}T}(A,B)=1$. Then, by (i) in Proposition 6.28 there exists $a\in A$ and $b\in B$ such that $a\leq b$, thus $\underline{a}\leq a\leq b\leq\overline{b}$. Conversely, let $\underline{a}\leq\overline{b}$. Since $\underline{a}\in A$, $\overline{b}\in B$, by (i) in Proposition 6.28, we immediately obtain $\mu_{\tilde{\leq}T}(A,B)=1$.

2. Observe that $\underline{a}\leq\overline{b}$ and $\underline{b}\leq\overline{a}$ are equivalent to the nonemptiness of $A\cap B$, i.e., there is c such that $c\in A\cap B$. However, by (i) in Proposition 6.28, this is equivalent to $\mu_{\tilde{=}T}(A,B)=1$.

(ii) 1. Let $\mu_{\tilde{\leq}_S}(A,B)=1$. Then, by (ii) in Proposition 6.28, for every $a\in A$ and every $b\in B$, we have $a\leq b$, thus $\overline{a}\leq\underline{b}$. Conversely, let $\underline{a}\leq\overline{a}\leq\underline{b}\leq\overline{b}$. Then, by (ii) in Proposition 6.28, we easily obtain $\mu_{\tilde{\leq}_S}(A,B)=1$.

2. Let $\mu_{\tilde{=}_S}(A,B)=1$. Then, by (ii) in Proposition 6.28, for every $a\in A$ and every $b\in B$, we have $a=b$, thus $\underline{a}=\overline{a}=\underline{b}=\overline{b}$. Conversely, let $\underline{a}=\overline{a}=\underline{b}=\overline{b}$. Then, by (ii) in Proposition 6.28, we obtain $\mu_{\tilde{=}_S}(A,B)=1$.

(iii) 1. Let $\mu_{\tilde{\leq}T,S}(A,B)=1$. Then, by (iii) in Proposition 6.28, there exists $a\in A$ such that for every $b\in B$ we have $a\leq b$, thus $\underline{a}\leq a\leq\underline{b}$. Conversely, let $\underline{a}\leq\underline{b}$. Then, by Proposition 6.28, (iii), we take $a=\underline{a}$ and since $\underline{b}\leq b$, we easily obtain $\mu_{\tilde{\leq}T,S}(A,B)=1$.

2. Let $\mu_{\tilde{=}T,S}(A,B)=1$. Then, by (iii) in Proposition 6.28, there exists $a\in A$ such that for every $b\in B$ we have $a=b$, thus $\underline{a}\leq\underline{b}=\overline{b}\leq\overline{a}$. Conversely, let $\underline{a}\leq\underline{b}=\overline{b}\leq\overline{a}$. Then, by (iii) in Proposition 6.28, we take $a=\underline{b}$ and immediately obtain $\mu_{\tilde{=}T,S}(A,B)=1$.

(iv) 1. Let $\mu_{\tilde{\leq}_{T,S}}(A,B)=1$. Then, by (iv) in Proposition 6.28, for every $b\in B$, there exists $a\in A$ such that $a\leq b$, thus $\underline{a}\leq a\leq\underline{b}$. Conversely, let $\underline{a}\leq\underline{b}$. Then, by (iv) in Proposition 6.28, we take $a=\underline{a}$ and since $\underline{b}\leq b$, we obtain $\mu_{\tilde{\leq}_{T,S}}(A,B)=1$.

2. Let $\mu_{\tilde{=}_{T,S}}(A,B)=1$. Then, (iv) in Proposition 6.28, for every $b\in B$, there exists $a\in A$ such that $a=b$, hence $\underline{a}\leq\underline{b}\leq\overline{b}\leq\overline{a}$. Conversely, let $\underline{a}\leq\underline{b}\leq\overline{b}\leq\overline{a}$. Then, by (iv) in Proposition 6.28, we take $a=\underline{b}$ and obtain $\mu_{\tilde{=}_{T,S}}(A,B)=1$.

(v) 1. Let $\mu_{\tilde{\leq}S,T}(A,B)=1$. Then, by (v) in Proposition 6.28, there exists $b\in B$ such that for every $a\in A$ we have $a\leq b$, thus $\overline{a}\leq b\leq\overline{b}$. Conversely, let $\overline{a}\leq\overline{b}$. Then, by (v) in Proposition 6.28, we take $b=\overline{b}$ and since $a\leq\overline{a}$ for every $a\in A$, we easily obtain $\mu_{\tilde{\leq}S,T}(A,B)=1$.

2. Let $\mu_{\tilde{=}S,T}(A,B)=1$. Then, by (v) in Proposition 6.28, there exists $b\in B$ such that for every $a\in A$ we have $a=b$, thus $\underline{b}\leq\underline{a}=\overline{a}\leq\overline{b}$.

Conversely, let $\underline{b} \leq \underline{a} = \overline{a} \leq \overline{b}$. Then, by (v) in Proposition 6.28, we take $b = \underline{a}$ and obtain $\mu_{\tilde{=}S,T}(A,B) = 1$.

(vi) 1. Let $\mu_{\tilde{\leq}_{S,T}}(A,B) = 1$. Then, by (vi) in Proposition 6.28, for every $a \in A$, there exists $b \in B$ such that $a \leq b$, thus $\overline{a} \leq b \leq \overline{b}$. Conversely, let $\overline{a} \leq \overline{b}$. Then, by (vi) in Proposition 6.28 , we take $b = \overline{b}$ and since $a \leq \overline{a}$, we therefore obtain $\mu_{\tilde{\leq}_{S,T}}(A,B) = 1$.

2. Let $\mu_{\tilde{=}_{S,T}}(A,B) = 1$. Then, by (vi) in Proposition 6.28, for every $a \in A$, there exists $b \in B$ such that $a = b$, thus $\underline{b} \leq \underline{a} \leq \overline{a} \leq \overline{b}$. Conversely, let $\underline{b} \leq \underline{a} \leq \overline{a} \leq \overline{b}$. Then, by (vi) in Proposition 6.28, we take $a = b$ and finally obtain $\mu_{\tilde{=}_{S,T}}(A,B) = 1$.

■

If A, B are nonempty closed intervals of $\mathbf{R}^m$, then by comparing (iii) and (iv) in Proposition 6.57, we can see that

$$\mu_{\tilde{\leq}T,S}(A,B) = 1 \text{ if and only if } \mu_{\tilde{\leq}_{T,S}}(A,B) = 1. \tag{6.57}$$

Likewise,

$$\mu_{\tilde{\leq}S,T}(A,B) = 1 \text{ if and only if } \mu_{\tilde{\leq}_{S,T}}(A,B) = 1, \tag{6.58}$$

as is clear from (v) and (vi), in the same proposition.

The following proposition shows that (6.57) and (6.58) can be presented in a stronger form.

PROPOSITION 6.58 *Let $A, B \in \mathcal{F}(\mathbf{R})$ be compact fuzzy sets, $T = \min$, $S = \max$. Then*

$$\mu_{\tilde{\leq}T,S}(A,B) = \mu_{\tilde{\leq}_{S,T}}(A,B), \mu_{\tilde{\leq}_{T,S}}(A,B) = \mu_{\tilde{\leq}S,T}(A,B).$$

The proof of Proposition 6.58 is given in [32] in a more general setting.

The following two propositions hold for a binary relation R on $X \times Y$, where X, Y are nonempty sets, and $\Psi^T(R)$ is the T-fuzzy extension of R.

PROPOSITION 6.59 *Let X, Y be sets, let $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$ be fuzzy sets given by the membership functions $\mu_A : X \to [0,1]$, $\mu_B : Y \to [0,1]$, respectively. Let T be a t-norm and R be a binary relation on $X \times Y$, $\Psi^T(R)$ be a T-fuzzy extension of R, let $\alpha \in (0,1)$.*

(i) *If $\mu_{\Psi^T(R)}(A,B) > \alpha$, then $\mu_{\Psi^T(R)}([A]_\alpha, [B]_\alpha) = 1$.*

(ii) *Let $T = \min$. If $\mu_{\Psi^T(R)}([A]_\alpha, [B]_\alpha) = 1$, then $\mu_{\Psi^T(R)}(A,B) \geq \alpha$.*

PROOF. (i) Let $\alpha \in (0,1)$, $\mu_{\Psi^T(R)}(A,B) > \alpha$. Then by (6.25) we obtain

$$\mu_{\Psi^T(R)}(A,B) = \sup\{T(\mu_A(u),\mu_B(v)) \mid uRv\}.$$

Since $\sup\{T(\mu_A(u),\mu_B(v)) \mid u \in X,\ v \in Y,\ uRv\} > \alpha$, there exist $u', v' \in \mathbf{R}^m$ such that $u'Rv'$ and $T(\mu_A(u'),\mu_B(v')) \geq \alpha$. Since T_M is the maximal t-norm, we have

$$T_M(\mu_A(u'),\mu_B(v')) \geq T(\mu_A(u'),\mu_B(v')) \geq \alpha.$$

Hence

$$\mu_A(u') \geq \alpha \text{ and } \mu_B(v') \geq \alpha,$$

in other words, $u' \in [A]_\alpha$, $v' \in [B]_\alpha$ and $u'Rv'$. By (i) in Proposition 6.28, we obtain

$$\mu_{\Psi^T(R)}([A]_\alpha,[B]_\alpha) = 1.$$

(ii) Let $T = \min$, $\mu_{\Psi^T(R)}([A]_\alpha,[B]_\alpha) = 1$. By Proposition 6.28 there exist $u'' \in [A]_\alpha$, $v'' \in [B]_\alpha$ such that $u''Rv''$. Then $\mu_A(u'') \geq \alpha$ and $\mu_B(v'') \geq \alpha$, therefore $\min\{\mu_A(u''),\mu_B(v'')\} \geq \alpha$. Consequently,

$$\mu_{\Psi^T(R)}(A,B) = \sup\{\min\{\mu_A(u),\mu_B(v)\} \mid u \in X,\ v \in Y,\ uRv\} \geq \alpha.$$

■

If we replace the strict inequality $>$ in (i) by $\geq$, then the conclusion of (i) is no longer true. To prove the result with $\geq$ instead of $>$, we assume that a condition similar to Condition (C) from the preceding section is satisfied.

PROPOSITION 6.60 *Let X, Y be sets, let $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$ be fuzzy sets given by the membership functions $\mu_A : X \to [0,1]$, $\mu_B : Y \to [0,1]$, respectively. Let T be a t-norm and R be a binary relation on $X \times Y$, $\Psi^T(R)$ be a T-fuzzy extension of R, let $\alpha \in (0,1]$. Suppose that there exist $u^* \in X, v^* \in Y$ such that u^*Rv^* and*

$$T(\mu_A(u^*),\mu_B(v^*)) = \sup\{T(\mu_A(u),\mu_B(v)) \mid u \in X,\ v \in Y,\ uRv\}. \tag{6.59}$$

If $\mu_{\Psi^T(R)}(A,B) \geq \alpha$, then $\mu_{\Psi^T(R)}([A]_\alpha,[B]_\alpha) = 1$.

PROOF. Let $\alpha \in (0,1]$, $\mu_{\tilde{R}^T}(A,B) \geq \alpha$. Then by (6.25) we obtain

$$\mu_{\Psi^T(R)}(A,B) = \sup\{T(\mu_A(u),\mu_B(v)) \mid u \in X,\ v \in Y,\ uRv\}. \tag{6.60}$$

Applying (6.59) and (6.60), we obtain $T(\mu_A(u^*),\mu_B(v^*)) \geq \alpha$. Since min is the maximal t-norm, we have

$$\min\{\mu_A(u^*),\mu_B(v^*)\} \geq T(\mu_A(u^*),\mu_B(v^*)) \geq \alpha,$$

hence

$$\mu_A(u^*) \geq \alpha \text{ and } \mu_B(v^*) \geq \alpha.$$

In other words, u^*Rv^* and $u^* \in [A]_\alpha$, $v^* \in [B]_\alpha$. By Proposition 6.28 we obtain

$$\mu_{\Psi^T(R)}([A]_\alpha, [B]_\alpha) = 1.$$

■

The following proposition gives some sufficient conditions for (6.59).

PROPOSITION 6.61 *Let A, B be compact fuzzy quantities with the membership functions $\mu_A : \mathbf{R}^m \to [0,1]$, $\mu_B : \mathbf{R}^m \to [0,1]$. Let T be a continuous t-norm and R be a closed binary relation on $\mathbf{R}^m$. Then there exist $u^*, v^* \in \mathbf{R}^m$ such that u^*Rv^* and*

$$T(\mu_A(u^*), \mu_B(v^*)) = \sup\{T(\mu_A(u), \mu_B(v)) \mid u, v \in \mathbf{R}^m,\ uRv\}.$$

PROOF. We show that $\varphi(u,v) = T(\mu_A(u), \mu_B(v))$ attains its maximum on the set $Z = \{(u,v) \in \mathbf{R}^{2m} \mid uRv\}$. Since R is closed binary relation, Z is a closed set. Further, since for all $\beta \in (0,1]$, the upper level sets $U(\varphi, \beta)$ are compact, it follows that either $U(\varphi, \beta) \cap Z$ is empty for all $\beta \in (0,1]$, or there exists $\beta_0 \in (0,1]$ such that $U(\varphi, \beta_0) \cap Z$ is nonempty.

In the former case, (6.59) holds for any $u^*, v^* \in Z$ with

$$T(\mu_A(u^*), \mu_B(v^*)) = 0.$$

In the latter case, there exists $(u^*, v^*) \in \mathbf{R}^{2m}$ such that φ attains its maximum on $U(\varphi, \beta_0) \cap Z$ at (u^*, v^*), which is a global maximizer of φ on Z. ■

COROLLARY 6.62 *Let $A, B \in \mathcal{F}(\mathbf{R}^m)$ be compact fuzzy quantities with the membership functions $\mu_A : \mathbf{R}^m \to [0,1]$, $\mu_B : \mathbf{R}^m \to [0,1]$. Let $T = \min$ and R be a closed binary relation on $\mathbf{R}^m$, $\Psi^T(R)$ be a T-fuzzy extension of R. For $\alpha \in (0,1]$,*

$$\mu_{\Psi^T(R)}(A, B) \geq \alpha \textit{ if and only if } \mu_{\Psi^T(R)}([A]_\alpha, [B]_\alpha) = 1.$$

PROOF. By Proposition 6.61, condition (6.59) is satisfied. Then by Proposition 6.60, we obtain the "if" part of the statement. The opposite part follows from Proposition 6.59. ■

Notice that the usual binary relations "=", "$\leq$" and "$\geq$" are closed binary relations.

Propositions 6.60, 6.59 and 6.61 hold for the T-fuzzy extension $\Psi^T(R)$ of the valued relation R. Similar results can be derived also for the other fuzzy extensions, particularly Ψ_S, $\Psi^{T,S}$, $\Psi_{T,S}$, $\Psi^{S,T}$ and $\Psi_{S,T}$. Here, we present an

important result for the particular case $m = 1$, i.e., $\mathbf{R}^m = \mathbf{R}$. The following theorem is a parallel to Theorem 6.57.

THEOREM 6.63 *Let $A, B \in \mathcal{F}(\mathbf{R})$ be strictly convex and compact fuzzy sets, $T = \min$, $S = \max$, $\alpha \in (0,1)$. Then*

(i) $\mu_{\Psi^T(\leq)}(A, B) \geq \alpha$ *if and only if* $\inf[A]_\alpha \leq \sup[B]_\alpha$.

(ii) $\mu_{\Psi_S(\leq)}(A, B) \geq \alpha$ *if and only if* $\sup[A]_{1-\alpha} \leq \inf[B]_{1-\alpha}$.

(iii) *The following are equivalent:*

- $\mu_{\Psi^{T,S}(\leq)}(A, B) \geq \alpha$,
- $\mu_{\Psi_{T,S}(\leq)}(A, B) \geq \alpha$,
- $\sup[A]_{1-\alpha} \leq \sup[B]_\alpha$.

(iv) *The following are equivalent:*

- $\mu_{\Psi^{S,T}(\leq)}(A, B) \geq \alpha$,
- $\mu_{\Psi_{S,T}(\leq)}(A, B) \geq \alpha$,
- $\inf[A]_\alpha \leq \inf[B]_{1-\alpha}$.

From the practical point of view, the last theorem is important for calculating the membership function of both fuzzy feasible solutions and fuzzy optimal solutions of fuzzy mathematical programming problem in Chapter 8. Some related results to this problem can be found also in [29].

II

APPLICATIONS

Mathematics is alive and well, but living under different names

—SIAM Report on Mathematics in Industry

Chapter 7

FUZZY MULTI-CRITERIA DECISION MAKING

1. Introduction

When dealing with practical decision problems, we often have to take into consideration uncertainty in the problem data. It may arise from errors in measuring physical quantities, from errors caused by representing some data in a computer, from the fact that some data are approximate solutions of other problems or estimations by human experts, etc. In some of these situations, the fuzzy set approach may be applicable. In the context of multicriteria decision making, functions mapping the set of feasible alternatives into the unit interval $[0, 1]$ of real numbers representing normalized utility functions can be interpreted as membership functions of fuzzy subsets of the underlying set of alternatives. However, functions with the range in $[0, 1]$ arise in more contexts.

In this chapter, we consider a *decision problem in* X, i.e., the problem to find a "best" decision in the set of feasible decisions X with respect to several criteria functions; see [100], [101], [102], [105], [106], [117]. Within the framework of such a decision situation, we deal with the existence and mutual relationships of three kinds of "optimal decisions": Weak Pareto-Maximizers, Pareto-Maximizers and Strong Pareto-Maximizers - particular alternatives satisfying some natural and rational conditions. We call them Pareto-Optimal decisions.

We study also the compromise decisions $x^* \in X$ maximizing some aggregation of given criteria μ_i, $i \in I = \{1, 2, \ldots, m\}$. The criteria μ_i considered here are functions defined on the set X of feasible decisions with the values in the unit interval $[0, 1]$, i.e., $\mu_i : X \to [0, 1]$, $i \in I$. Such functions can be interpreted as membership functions of fuzzy subsets of X and will be called here *fuzzy criteria*. Later on, in Chapters 8 and 9, each constraint or objective function of the fuzzy mathematical programming problem will be naturally as-

sociated with a unique fuzzy criterion. From this point of view this chapter should follow the Chapters 8 and 9 dealing with fuzzy mathematical programming. Our approach here is, however, more general and can be adopted to a more general class of decision problems.

The set X of feasible decisions is supposed to be a convex subset or a generalized convex subset of the n-dimensional Euclidean space $\mathbf{R}^n$, frequently we consider $X = \mathbf{R}^n$. The main subject of our interest is to derive relations between Pareto-optimal decisions and compromise decisions. Moreover, we generalize the concept of the compromise decision by adopting aggregation operators which were investigated in Chapter 5, and we also extend the results derived for max-min decisions. The results will be derived for the n-dimensional Euclidean vector space $\mathbf{R}^n$ with $n \geq 1$. However, some results can be derived only for $\mathbf{R}^1$, denoted here simply by $\mathbf{R}$.

2. Fuzzy Criteria

Since no function mapping $\mathbf{R}$ into $[0, 1]$ is strictly concave, and each concave function mapping $\mathbf{R}$ into $[0, 1]$ is constant on $\mathbf{R}$, we take advantage of Definition 3.1 in Chapter 3, where we defined (quasi)concave and (quasi)-convex functions on arbitrary subsets X of $\mathbf{R}^n$. Now, for membership functions of fuzzy subsets of $\mathbf{R}^n$, the concavity concepts defined in Definition 3.1 will be applied to the supports of the membership functions, that is, for a fuzzy subset A of $\mathbf{R}^n$ with the membership function $\mu : \mathbf{R}^n \to [0, 1]$, we consider $X = \mathrm{Supp}(A)$. We use the notation and nomenclature of Chapter 6. However, we will sometimes use also the notation introduced in Chapter 4, Definition 4.26, namely, $\mathrm{Supp}(\mu)$. Likewise, we will use $\mathrm{Core}(A)$ and $\mathrm{Core}(\mu)$ in the same meaning, provided μ is a membership function of fuzzy subset A.

It is evident that a function $\mu : \mathbf{R}^n \to [0, 1]$ quasiconcave on $\mathbf{R}^n$ is quasiconcave on $\mathrm{Supp}(\mu)$, and vice versa: any function μ quasiconcave on $\mathrm{Supp}(\mu)$ is quasiconcave on $\mathbf{R}^n$. However, this is no longer true that the membership functions strictly (quasi)concave on $\mathbf{R}^n$ and membership functions strictly (quasi)concave on their supports coincide.

EXAMPLE 7.1 Let $\mu : \mathbf{R} \to [0, 1]$ be defined by $\mu(x) = \max\{0, 1 - x^2\}$, see Figure 7.1(a). It can easily be shown that $\mathrm{Supp}(\mu) = [-1, 1]$ and μ is strictly concave and strictly quasiconcave on $\mathrm{Supp}(\mu)$. However, μ is neither strictly concave nor strictly quasiconcave on $\mathbf{R}$. In Figure 7.1 (b), a semistrictly quasiconcave function on $\mathrm{Supp}(\mu)$, which is not semistrictly quasiconcave on $\mathbf{R}$, is depicted. □

As mentioned in the introduction, we are interested in the properties of solution concepts of optimization problems whose objectives are expressed in the terms of fuzzy criteria. A particular interest will be given to fuzzy criteria defined as follows.

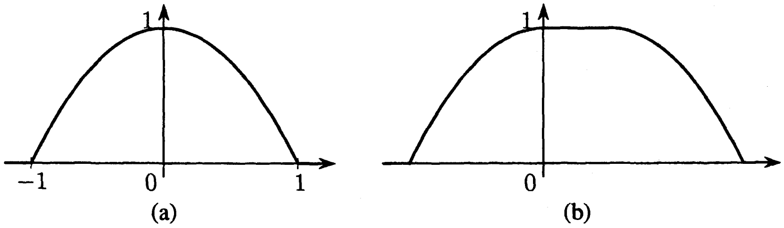

Figure 7.1.

DEFINITION 7.2 *A fuzzy subset of* $\mathbf{R}^n$ *given by its membership function* $\mu : \mathbf{R}^n \to [0,1]$ *is called a* fuzzy criterion on $\mathbf{R}^n$ *if* μ *is upper normalized.*

By Definition 7.2 any fuzzy criterion is given by the membership function attaining the maximal membership value 1. Sometimes in this chapter, the results will be derived for membership functions of fuzzy subsets of $\mathbf{R}^n$ not necessarily upper normalized. This fact will be explicitly mentioned if necessary.

Fuzzy criteria in one-dimensional Euclidean space $\mathbf{R}$ with additional concavity properties can be characterized by the following simple propositions. The corresponding proofs follow easily from the definition.

PROPOSITION 7.3 *If the membership function* μ *of a fuzzy criterion on* $\mathbf{R}$ *is quasiconcave on* $\mathbf{R}$*, then there exist* $\alpha, \beta \in [0,1]$ *and* $a, b, c, d \in \mathbf{R} \cup \{-\infty, +\infty\}$*, such that* $a \le b \le c \le d$ *and*

- $\mu(x) = \alpha$ *for* $x < a$,
- $\mu(x)$ *is non-decreasing for* $a \le x \le b$,
- $\mu(x) = 1$ *for* $b < x < c$,
- $\mu(x)$ *is non-increasing for* $c \le x \le d$,
- $\mu(x) = \beta$ *for* $d < x$.

PROPOSITION 7.4 *If the membership function* μ *of a fuzzy criterion on* $\mathbf{R}$ *is strictly quasiconcave on* Supp(μ) *and* Supp(μ) *is convex, then there exist* $a, b \in \mathbf{R} \cup \{-\infty, +\infty\}$*, and* $\bar{x} \in \mathbf{R}$*, such that* $a \le \bar{x} \le b$ *and*

- $\mu(x) = 0$ *for* $x \le a$ *or* $x \ge b$,
- $\mu(x)$ *is increasing for* $a \le x \le \bar{x}$,
- $\mu(\bar{x}) = 1$,

- *$\mu(x)$ is decreasing for $\bar{x} \leq x \leq b$.*

Later on, we shall take advantage of the above stated properties in case of $\mathbf{R}^n$ for $n > 1$.

3. Pareto-Optimal Decisions

Throughout this chapter we suppose that $I = \{1, 2, \ldots, m\}$, $m > 1$, I is an index set of a given family $F = \{\mu_i \mid i \in I\}$ of membership functions of fuzzy subsets of $\mathbf{R}^n$. Let X be a subset of $\mathbf{R}^n$ such that Supp$(\mu_i) \subset X$ for all $i \in I$. The elements of X are called *decisions*.

DEFINITION 7.5

(i) *A decision x_{WP} is said to be a* Weak Pareto-Maximizer (WPM), *if there is no $x \in X$ such that*

$$\mu_i(x_{WP}) < \mu_i(x), \quad \textit{for every } i \in I.$$

(ii) *A decision x_P is said to be a* Pareto-Maximizer (PM), *if there is no $x \in X$ such that*

$$\mu_i(x_P) \leq \mu_i(x), \quad \textit{for every } i \in I,$$
$$\mu_i(x_P) < \mu_i(x), \quad \textit{for some } i \in I.$$

(iii) *A decision x_{SP} is said to be a* Strong Pareto-Maximizer (SPM), *if there is no $x \in X$, $x \neq x_{SP}$, such that*

$$\mu_i(x_{SP}) \leq \mu_i(x), \quad \textit{for every } i \in I.$$

DEFINITION 7.6 *The sets of all WPM, PM and SPM are denoted by X_{WP}, X_P, X_{SP}, respectively. The elements of $X_{WP} \cup X_P \cup X_{SP}$ are called* Pareto-Optimal Decisions.

The following property is evident.

PROPOSITION 7.7 *Any SPM is PM, and any PM is WPM, i.e.,*

$$X_{SP} \subset X_P \subset X_{WP}. \tag{7.1}$$

To illustrate the above concepts, let us inspect the following example.

EXAMPLE 7.8 Let μ_1 and μ_2 be as in Figure 7.2. Then $X_{WP} = [a, g]$, $X_P = [b, c] \cup (d, e) \cup \{f\}$, $X_{SP} = (d, e) \cup \{f\}$. □

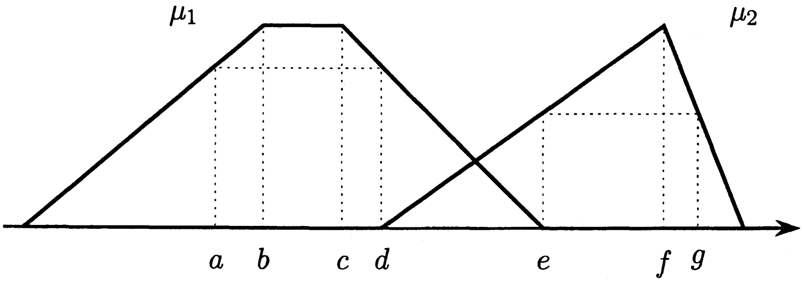

Figure 7.2.

Let $D_j \subset \mathbf{R}^n$ for all $j \in J$, where J is a finite index set. By $\mathrm{Conv}\{D_j \mid j \in J\}$ we denote the *convex hull* of all sets D_j, i.e.,

$$\mathrm{Conv}\{D_j \mid j \in J\} = \left\{ z \in \mathbf{R}^n \,\middle|\, z = \textstyle\sum_{j \in J} \lambda_j x_j,\ x_j \in D_j,\ \lambda_j \geq 0,\ \sum_{j \in J} \lambda_j = 1 \right\}.$$

In the following propositions, we obtain a transparent characterization of all WPM, PM and SPM for strictly quasiconcave fuzzy criteria in one-dimensional space $\mathbf{R}$. Unfortunately, we cannot obtain parallel results in $\mathbf{R}^n$ for $n > 1$, as is demonstrated by Example 7.10; see also [106].

PROPOSITION 7.9 *Let μ_i, $i \in I$, be membership functions of fuzzy criteria on $\mathbf{R}$. If each μ_i is quasiconcave on $\mathbf{R}$, then*

$$\mathrm{Conv}\{\mathrm{Core}(\mu_i) \mid i \in I\} \subset X_{WP}.$$

PROOF. Let $x'_i = \inf \mathrm{Core}(\mu_i)$, $x''_i = \sup \mathrm{Core}(\mu_i)$ and set $x' = \min\{x'_i \mid i \in I\}$, $x'' = \max\{x''_i \mid i \in I\}$, then $\mathrm{Cl}(\mathrm{Conv}\{\mathrm{Core}(\mu_i) \mid i \in I\}) = [x', x'']$. Let $x \in \mathrm{Conv}\{\mathrm{Core}(\mu_i) \mid i \in I\}$ and suppose that $x \notin X_{WP}$. Then there exists y with $\mu_i(y) > \mu_i(x)$, for all $i \in I$. Assume that $y < x$, then there exists $k \in I$ and $y' \in \mathrm{Core}(\mu_k)$ with $y < x \leq y'$ such that by Proposition 7.3 we obtain $\mu_k(y) \leq \mu_k(x)$, a contradiction. Otherwise, if $x < y$, then again by Proposition 7.3 we have $\mu_j(x) \geq \mu_j(y)$, again a contradiction. ∎

EXAMPLE 7.10 This example demonstrates that Proposition 7.9 is not true for $\mathbf{R}^n$, where $n > 1$, particularly for $n = 2$. Set

$$\begin{aligned}\mu_1(x_1, x_2) &= \max\left\{0, 1 - \frac{1}{4}x_1^2 - x_2^2\right\},\\ \mu_2(x_1, x_2) &= \max\left\{0, 1 - (x_1 - 1)^2 - (x_2 - 1)^2\right\}.\end{aligned}$$

Notice that μ_1, μ_2 are continuous membership functions of fuzzy criteria, strictly concave on their supports, hence quasiconcave on $\mathbf{R}^2$. Here, $\mathrm{Core}(\mu_1) = \bar{\mathbf{x}}_1 = (0,0)$, $\mathrm{Core}(\mu_2) = \bar{\mathbf{x}}_2 = (1,1)$ are the end points of the segment $\mathrm{Conv}\{\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2\}$ in $\mathbf{R}^2$. It is easy to calculate that

$$\mu_1(0.5, 0.5) = 0.6875 \text{ and } \mu_2(0.5, 0.5) = 0.5.$$

On the other hand, $\mu_1(0.7, 0.4) = 0.7175$, $\mu_2(0.7, 0.4) = 0.55$. We obtain $\mu_1(0.5, 0.5) < \mu_1(0.7, 0.4)$ and $\mu_2(0.5, 0.5) < \mu_2(0.7, 0.4)$. As $(0.5, 0.5) \in \mathrm{Conv}\{\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2\}$ and $(0.7, 0.4) \notin \mathrm{Conv}\{\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2\}$, it follows that

$$\mathrm{Conv}\{\mathrm{Core}(\mu_i) \mid i = 1, 2\} = \mathrm{Conv}\{\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2\},$$

which is not a subset of X_{WP}. □

EXAMPLE 7.11 Let μ_1, μ_2 be as in Figure 7.3. Here, μ_1, μ_2 are continuous and evidently $X_{WP} = [a, d]$, $X_P = [b, c]$, however, $X_{SP} = \emptyset$. □

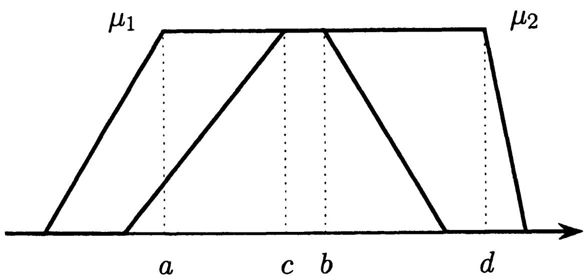

Figure 7.3.

PROPOSITION 7.12 *Let μ_i, $i \in I$, be membership functions of fuzzy criteria on $\mathbf{R}$, strictly quasiconcave on their convex supports. Then*

$$X_P \subset \mathrm{Conv}\{\mathrm{Core}(\mu_i) \mid i \in I\}.$$

PROOF. By Proposition 7.4, each $\mathrm{Core}(\mu_i)$ contains exactly one element, i.e., $x_i = \mathrm{Core}(\mu_i)$. Setting $x' = \min\{x_i \mid i \in I\}$, $x'' = \max\{x_i \mid i \in I\}$, we have

$$\mathrm{Conv}\{\mathrm{Core}(\mu_i) \mid i \in I\} = [x', x''].$$

Let $x \notin \mathrm{Conv}\{\mathrm{Core}(\mu_i) \mid i \in I\}$ and suppose that $x < x'$. By monotonicity of μ_i we get $\mu_i(x) \leq \mu_i(x')$ for all $i \in I$. Moreover, by Proposition 7.4, $\mu_j(x) < \mu_j(x')$ for j satisfying $x_j = x'$, consequently $x \notin X_P$.

On the other hand, suppose that $x'' < x$. Again by monotonicity of μ_i we get $\mu_i(x'') \geq \mu_i(x)$ for all $i \in I$, and by Proposition 7.4, $\mu_k(x'') > \mu_k(x)$ for k satisfying $x_k = x''$. Hence, again $x \notin X_P$, which gives the required result. ∎

PROPOSITION 7.13 *Let μ_i, $i \in I$, be membership functions of fuzzy criteria on* **R**, *strictly quasiconcave on their convex supports. If*

$$\mathrm{Conv}\{\mathrm{Core}(\mu_i) \mid i \in I\} \subset \bigcap_{i \in I} \mathrm{Supp}(\mu_i), \tag{7.2}$$

then

$$X_{WP} = X_P = X_{SP} = \mathrm{Conv}\{\mathrm{Core}(\mu_i) \mid i \in I\}. \tag{7.3}$$

PROOF. By Proposition 7.4, each $\mathrm{Core}(\mu_i)$ contains exactly one element, i.e., $x_i = \mathrm{Core}(\mu_i)$. Setting $x' = \min\{x_i \mid i \in I\}$, $x'' = \max\{x_i \mid i \in I\}$, we obtain

$$\mathrm{Conv}\{\mathrm{Core}(\mu_i) \mid i \in I\}) = [x', x''].$$

First, we prove that $[x', x''] \subset X_{SP}$.

Let $x \in [x', x'']$ and suppose by contrary that $x \notin X_{SP}$. Then there exists y with $y \neq x$ and $\mu_i(y) \geq \mu_i(x)$, for all $i \in I$.

Further, suppose that $y < x$, then by strict quasiconcavity of μ_k, for k satisfying $x_k = x''$, and by Proposition 7.4, we get $\mu_k(y) < \mu_k(x)$, a contradiction.

On the other hand, if $x < y$, then by strict quasiconcavity of μ_j, for j satisfying $x_j = x'$, we get $\mu_j(x) > \mu_j(y)$, again a contradiction. Hence

$$\mathrm{Conv}\{\mathrm{Core}(\mu_i) \mid i \in I\}) \subset X_{SP}.$$

Second, we prove that $X_{WP} \subset \mathrm{Conv}\{\mathrm{Core}(\mu_i) \mid i \in I\}$.

Suppose that $y \notin [x', x'']$. Let $y < x'$. From (7.2), it follows $\mu_i(x') > 0$ for all $i \in I$. Applying strict monotonicity of μ_i, we get $\mu_i(y) < \mu_i(x')$, for all $i \in I$, hence, $y \notin X_{WP}$. Assuming $y > x''$, we obtain by analogy the same result.

Combining the first and the second result, we obtain the chain of inclusions

$$X_{WP} \subset \mathrm{Conv}\{\mathrm{Core}(\mu_i) \mid i \in I\} \subset X_{SP}.$$

However, by (7.1) we have $X_{SP} \subset X_P \subset X_{WP}$; consequently, we obtain the required equalities (7.3). ∎

Notice that inclusion (7.2) is satisfied if all $\mathrm{Supp}(\mu_i)$, $i \in I$, are identical.

4. Compromise Decisions

In the theory of multi-objective optimization, the "compromise decision" or "compromise solution" is obtained as the solution of a single-objective problem with the objective being a combination of all criteria in question; see, e.g., [56].

In this section we investigate a concept and some properties of compromise decision $x^* \in X$, maximizing the aggregation of all criteria, e.g., $\min\{\mu_i(x) \mid i \in I\}$, where $X \subset \mathbf{R}^n$ is a convex set, $I = \{1, 2, \ldots, m\}$, $m > 1$. The original idea belongs to Bellman and Zadeh in [11], who proposed its use in decision analysis by using the following definition.

DEFINITION 7.14 *Let X be a convex subset of $\mathbf{R}^n$ and let μ_i, $i \in I$, be the membership functions of fuzzy subsets of X. A decision $x^* \in X$ is called a* max-min decision, *if*

$$\min\{\mu_i(x^*) \mid i \in I\} = \max\{\min\{\mu_i(x) \mid i \in I\} \mid x \in X\}.$$

The set of all max-min decisions in X is denoted by X_M.

We start with two propositions that are concerned with the existence of max-min decision. In $\mathbf{R}$, the requirement of compactness of the α-cuts of the criteria is not necessary for the existence of nonempty X_M, whereas the same result is no longer true in $\mathbf{R}^n$ with $n > 1$, which will be demonstrated on an example.

PROPOSITION 7.15 *Let μ_i, $i \in I$, be the membership functions of fuzzy criteria on $\mathbf{R}$, quasiconcave on $\mathbf{R}$. If all μ_i are upper semicontinuous (USC) on $\mathbf{R}$, then $X_M \neq \emptyset$.*

PROOF. For each $i \in I$ there exists $x_i \in \mathrm{Core}(\mu_i)$. Put $x' = \min\{x_i \mid i \in I\}$ and $x'' = \max\{x_i \mid i \in I\}$. By Proposition 7.3, μ_i are non-decreasing in $(-\infty, x']$ and non-increasing in $[x'', +\infty)$ for all $i \in I$. Then $\varphi = \min\{\mu_i \mid i \in I\}$ is also non-decreasing in $(-\infty, x']$ and non-increasing in $[x'', +\infty)$. As μ_i, $i \in I$, are upper semicontinuous, φ is also USC on $\mathbf{R}$, particularly on the compact interval $[x', x'']$. Hence $\varphi = \min\{\mu_i \mid i \in I\}$ attains its maximum on $[x', x'']$ at a global maximizer on $\mathbf{R}$. This maximizer is a max-min decision, i.e., $X_M \neq \emptyset$. ∎

EXAMPLE 7.16 This example demonstrates that semicontinuity is essential in the above proposition. Let

$$\mu_1(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x \geq 0, \end{cases} \qquad \mu_2(x) = \begin{cases} e^x & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Here, μ_1, μ_2 are the membership functions of fuzzy criteria, μ_2 is continuous, μ_1 is not upper semicontinuous on $\mathbf{R}$. It is easy to see that $\psi(x) = \min\{\mu_1(x), \mu_2(x)\}$ does not attain its maximum on $\mathbf{R}$, i.e., $X_M = \emptyset$. □

EXAMPLE 7.17 This example demonstrates that Proposition 7.15 does not hold in $\mathbf{R}^n$ with $n > 1$, particularly with $n = 2$. Set

$$\mu_1(x_1, x_2) = \max\left\{0, \min\left\{x_2, 1 - \left(\frac{x_1 + 1}{x_2 + 1}\right)^2\right\}\right\},$$

$$\mu_2(x_1, x_2) = \max\left\{0, \min\left\{x_2, 1 - \left(\frac{x_1 - 1}{x_2 + 1}\right)^2\right\}\right\}.$$

It can easily be verified that μ_1 and μ_2 are continuous fuzzy criteria on $\mathbf{R}^2$, quasiconcave on $\mathbf{R}^2$. Let

$$\varphi(x_1, x_2) = \min\{\mu_1(x_1, x_2), \mu_2(x_1, x_2)\}.$$

It is not difficult to show that $\varphi(x_1, x_2) < 1$ on $\mathbf{R}^2$ and $\varphi(0, x_2) = 1 - 1/(x_2 + 1)^2$ for $x_2 > 1$. Since $\lim \varphi(0, x_2) = 1$ for $x_2 \to +\infty$, φ does not attain its maximum on X, i.e., $X_M(\mu_1, \mu_2) = \emptyset$. □

Recall that a fuzzy subset A given by the membership function $\mu : \mathbf{R}^n \to [0, 1]$ is compact if and only if μ is USC on $\mathbf{R}^n$ and $[A]_\alpha$ is a bounded subset of $\mathbf{R}^n$ for every $\alpha \in (0, 1]$, or, $[A]_\alpha$ is a compact subset of $\mathbf{R}^n$ for every $\alpha \in (0, 1]$.

PROPOSITION 7.18 *Let A be a compact fuzzy subset of $\mathbf{R}^n$ given by the membership function $\mu : \mathbf{R}^n \to [0, 1]$. Then μ attains its maximum on $\mathbf{R}^n$.*

PROOF. Let $\alpha^* = \sup\{\mu(x) \mid x \in \mathbf{R}^n\} > 0$ and let $\{\alpha_k\}_{k=1}^{\infty}$ is an increasing sequence of numbers such that $\alpha_k \in (0, 1)$ and $\alpha_k \to \alpha^*$. (If $\alpha^* = 0$, then there is nothing to prove.) Then $[A]_{\alpha_k}$ is compact and $[A]_{\alpha_{k+1}} \subset [A]_{\alpha_k}$ for all $k = 1, 2, \ldots$. By the well known property of compact sets there exists $x^* \in \mathbf{R}^n$ with

$$x^* \in \bigcap_{k=1}^{\infty} [A]_{\alpha_k}. \tag{7.4}$$

It remains to show that

$$\mu(x^*) = \alpha^*. \tag{7.5}$$

On contrary, suppose that $\mu(x^*) < \alpha^*$. Then there exists k_0 such that

$$\mu(x^*) < \alpha_{k_0} \leq \alpha^*. \tag{7.6}$$

By (7.4), $x^* \in [A]_{\alpha_{k_0}}$, hence, $\mu(x^*) \geq \alpha_{k_0}$, a contradiction to (7.6). Consequently, (7.5) is true, which completes the proof. ■

PROPOSITION 7.19 *Let μ_i , $i \in I$, be membership functions of fuzzy subsets A_i of $\mathbf{R}^n$. If A_i is compact for every $i \in I$, then $X_M \neq \emptyset$.*

PROOF. Let $\varphi = \min\{\mu_i \mid i \in I\}$. Observe that $[\varphi]_\alpha = \bigcap_{i\in I}[\mu_i]_\alpha$ for all $\alpha \in (0,1]$. Since all $[\mu_i]_\alpha$ are compact, the same holds for $[\varphi]_\alpha$. By Proposition 7.18, φ attains its maximum on $\mathbf{R}^n$, i.e., $X_M \neq \emptyset$. ■

In what follows we investigate some relationships between Pareto-Optimal decisions and max-min decisions. In the following two propositions, normality of μ_i is not required.

PROPOSITION 7.20 *Let μ_i, $i \in I$, be membership functions of fuzzy subsets of $\mathbf{R}^n$. Then*

$$X_M \subset X_{WP}.$$

PROOF. Let $x^* \in X_M$. Suppose that x^* is not a Weak Pareto-Maximizer, then by Definition 7.5 there exists x' such that $\mu_i(x^*) < \mu_i(x')$, for all $i \in I$. Then

$$\min\{\mu_i(x^*) \mid i \in I\} < \min\{\mu_i(x') \mid i \in I\},$$

which shows that x^* is not a max-min decision, a contradiction. ■

PROPOSITION 7.21 *Let μ_i, $i \in I$, be the membership functions of fuzzy sets of $\mathbf{R}^n$, let $X_M = \{x^*\}$, i.e. $x^* \in X$ be a unique max-min decision. Then*

$$X_M \subset X_{SP}.$$

PROOF. Suppose that x^* is a unique max-min decision and suppose that x^* is not a SPM. Then there exists $x^+ \in X$, $x^* \neq x^+$, $\mu_i(x^*) \leq \mu_i(x^+)$, for all $i \in I$. Then

$$\min\{\mu_i(x^*) \mid i \in I\} \leq \min\{\mu_1(x^+) \mid i \in I\},$$

which is a contradiction with the uniqueness of $x^* \in X_M$. ■

In the following proposition sufficient conditions for the uniqueness of a compromise decision are given.

PROPOSITION 7.22 *Let μ_i, $i \in I$, be membership functions of fuzzy subsets of $\mathbf{R}^n$ that are strictly quasiconcave on their convex supports* $\mathrm{Supp}(\mu_i)$. *Let $x^* \in X_M$ be such that*

$$\min\{\mu_i(x^*) \mid i \in I\} > 0. \tag{7.7}$$

Then $X_M = \{x^*\}$, i.e., the max-min decision x^* is unique.

PROOF. Let $\varphi(x) = \min\{\mu_i(x) \mid i \in I\}$, where $x \in \mathbf{R}^n$, and suppose that there exists $x' \in X_M$, $x^* \neq x'$. Then by (7.7)

$$\varphi(x') = \varphi(x^*) > 0.$$

As by (7.7) we have also $x^*, x' \in X = \bigcap_{i\in I} \text{Supp}(\mu_i)$, where X is convex. By strict quasiconcavity of φ on X we obtain for $x^+ = \lambda x' + (1-\lambda)x^*$ and $0 < \lambda < 1$:

$$\varphi(x^+) > \min\{\varphi(x'), \varphi(x^*)\},$$

which contradicts the fact that $\varphi(x^*) = \max\{\varphi(x) \mid x \in \mathbf{R}^n\}$. Consequently, X_M consists of one element only. ■

COROLLARY 7.23 *Let $\mu_i, i \in I$, be membership functions of fuzzy subsets of $\mathbf{R}^n$. If each μ_i is strictly quasiconcave on its convex support and $x^* \in X_M$ satisfies (7.7), then $x^* \in X_{SP}$.*

5. Generalized Compromise Decisions

In this section we generalize the concept of the max-min decision by adopting aggregation operators investigated in Chapter 5.

DEFINITION 7.24 *Let X be a convex subset of $\mathbf{R}^n$ and let μ_i, $i \in I$, be the membership functions of fuzzy subsets of X. Let $G = \{G_m\}_{m=1}^{\infty}$ be an aggregation operator. A decision $x^* \in X$ is called a* max-G decision, *if*

$$G_m(\mu_1(x^*), \ldots, \mu_m(x^*)) = \max\{G_m(\mu_1(x), \ldots, \mu_m(x)) \mid x \in X\}.$$

The set of all max-G decisions on X is denoted by $X_G(\mu_1, \ldots, \mu_m)$, or, shortly, X_G.

If there is no danger of misunderstanding, we omit the subscript m in the aggregating mapping G_m, writing shortly G. In the previous section we have investigated some properties of the compromise decisions considering a particular aggregation operator, namely the t-norm T_M. In what follows we extend the results from the previous section to more general aggregation operators. The following propositions generalize Propositions 7.19, 7.20, 7.21 and 7.22.

PROPOSITION 7.25 *Let $\mu_i, i \in I$, be the USC membership functions of fuzzy subsets of $\mathbf{R}^n$, G be an USC aggregation operator. Then $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by*

$$\psi(x) = G(\mu_1(x), \ldots, \mu_m(x)) \tag{7.8}$$

is USC on $\mathbf{R}^n$.

PROOF. Let $x_0 \in R^n$ and $\varepsilon > 0$. It is sufficient to prove that there exist $\delta > 0$, such that $\psi(x) \leq \psi(x_0) + \varepsilon$ for every $x \in B(x_0, \delta) = \{x \in R^n \mid \|x - x_0\| < \delta\}$.

Let $y_{0i} = \mu_i(x_0)$ for $i \in I$, put $y_0 = (y_{01}, \ldots, y_{0m})$. Since G is USC on $[0,1]^m$, there exists $\eta > 0$, such that $y \in B(y_0, \eta) = \{y \in [0,1]^m \mid \|y - y_0\| < \eta\}$ implies

$$G(y_1, \ldots, y_m) \leq G(y_{01}, \ldots, y_{0m}) + \varepsilon. \tag{7.9}$$

By upper semicontinuity of all μ_i, $i \in I$, there exists $\delta > 0$ such that $\mu_i(x) \leq \mu_i(x_0) + \eta/2$ for every $x \in B(x_0, \delta)$. By monotonicity property of G, we obtain

$$G(\mu_1(x), \ldots, \mu_m(x)) \leq G(z_1, \ldots, z_m), \tag{7.10}$$

where $z_i = \min\{1, \mu_i(x_0) + \eta/2\}$, and also $(z_1, ..., z_m) \in B(y_0, \eta)$. Moreover, by (7.8) we have $\psi(x_0) = G(y_{01}, \ldots, y_{0m})$. Combining inequalities (7.9) and (7.10), we obtain the required result $\psi(x) \leq \psi(x_0) + \varepsilon$. ∎

The next two proposition give some sufficient conditions for the existence of max-G decisions.

PROPOSITION 7.26 *Let μ_i, $i \in I$, be membership functions of fuzzy subsets A_i of $\mathbf{R}^n$, G be an USC and idempotent aggregation operator. If A_i is compact for every $i \in I$, then $X_G \neq \emptyset$.*

PROOF. Let $\alpha \in (0,1]$, $\psi(x) = G(\mu_1(x), \ldots, \mu_m(x))$. We prove that $[\psi]_\alpha = \{x \in R^n \mid G(\mu_1(x), \ldots, \mu_m(x)) \geq \alpha\}$ is a compact subset of $\mathbf{R}^n$. First, we prove that $[\psi]_\alpha$ is bounded. Assume the contrary; then there exist $x_k \in [\psi]_\alpha$, $k = 1, 2, \ldots$, with $\lim \|x_k\| = +\infty$ for $k \to +\infty$. Take an arbitrary β, with $0 < \beta < \alpha$. Since all $[\mu_i]_\beta$ are bounded, then there exists k_0 such that for all $i \in I$ and $k > k_0$ we obtain $x_k \notin [\mu_i]_\beta$, i.e., $\mu_i(x_k) < \beta$. By monotonicity and idempotency of A it follows that for $k > k_0$ we have

$$G(\mu_1(x_k), \ldots, \mu_m(x_k)) \leq G(\beta, \ldots, \beta) = \beta < \alpha.$$

But this is a clear contradiction, consequently, $[\psi]_\alpha$ is bounded.

By Proposition 7.25 $\psi(x) = G(\mu_1(x), \ldots, \mu_m(x))$ is USC on $\mathbf{R}^n$, hence $[\psi]_\alpha$ is closed. Consequently, $[\psi]_\alpha$ is compact. Then ψ is a membership function of a compact fuzzy subset of $\mathbf{R}^n$, therefore, by Proposition 7.18, ψ attains its maximum on $\mathbf{R}^n$, i.e., $X_G \neq \emptyset$. ∎

Notice that the mean aggregation operators (5.1) - (5.8) are idempotent. However, by Proposition 5.3, the only idempotent t-norm is the minimum t-norm T_M. The following proposition extends the statement of Proposition 7.26 to all other USC t-norms.

PROPOSITION 7.27 *Let μ_i, $i \in I$, be membership functions of fuzzy subsets A_i of $\mathbf{R}^n$, T be an USC t-norm. If A_i is compact for every $i \in I$, then $X_T \neq \emptyset$.*

PROOF. Let $\phi : \mathbf{R}^n \rightarrow [0,1]$ be defined for $x \in \mathbf{R}^n$ as

$$\phi(x) = T(\mu_1(x), \ldots, \mu_m(x)).$$

Let $\alpha \in (0,1]$. We prove that $[\phi]_\alpha$ is a bounded subset of $\mathbf{R}^n$. By definition we have $[\phi]_\alpha = \{x \in R^n \mid T(\mu_1(x), \ldots, \mu_m(x)) \geq \alpha\}$. Since $[\mu_i]_\alpha$ are bounded for all $i \in I$, it follows that

$$\bigcap_{i \in I} [\mu_i]_\alpha = \{x \in R^n \mid \min\{\mu_1(x), \ldots, \mu_m(x)\} \geq \alpha\}$$

is also bounded. We show that $[\phi]_\alpha \subset \bigcap_{i \in I} [\mu_i]_\alpha$.

Let $x \in [\phi]_\alpha$. Then by definition of α-cut we have

$$T(\mu_1(x), \ldots, \mu_m(x)) \geq \alpha. \tag{7.11}$$

Since T is dominated by the minimum t-norm T_M, it follows that

$$T(\mu_1(x), \ldots, \mu_m(x)) \leq \min\{\mu_1(x), \ldots, \mu_m(x)\}. \tag{7.12}$$

From (7.11) and (7.12) we obtain

$$\alpha \leq \min\{\mu_1(x), \ldots, \mu_m(x)\},$$

thus $x \in \bigcap_{i \in I} [\mu_i]_\alpha$, proving that $[\phi]_\alpha \subset \bigcap_{i \in I} [\mu_i]_\alpha$. Consequently, $[\phi]_\alpha$ is bounded for all $\alpha \in (0,1]$.

By Proposition 7.25 $\phi(x) = T(\mu_1(x), \ldots, \mu_m(x))$ is USC on $\mathbf{R}^n$ and by Proposition 7.18 ϕ attains its maximum on $\mathbf{R}^n$, i.e., $X_T \neq \emptyset$. ■

The following proposition is an extension of Proposition 7.20.

PROPOSITION 7.28 *Let μ_i , $i \in I$, be the membership functions of fuzzy subsets of $\mathbf{R}^n$, G be a strictly monotone aggregation operator. Then*

$$X_G \subset X_{WP}.$$

PROOF. Let $x^* \in X_G$. Suppose that x^* is not a Weak Pareto-Maximizer, then by Definition 4.27 there exists x' such that $\mu_i(x^*) < \mu_i(x')$, for all $i \in I$. Then by the strict monotonicity of G we obtain

$$G(\mu_1(x^*), \ldots, \mu_m(x^*)) < G(\mu_1(x'), \ldots, \mu_m(x')),$$

showing that x^* is not a max-G decision, a contradiction. ■

The following proposition is a generalization of Proposition 7.21.

PROPOSITION 7.29 *Let μ_i , $i \in I$, be the membership functions of fuzzy subsets of $\mathbf{R}^n$, let $X_G = \{x^*\}$, i.e., $x^* \in \mathbf{R}^n$ be a unique max-G decision, let G be an aggregation operator. Then*

$$X_G \subset X_{SP}.$$

PROOF. Suppose that x^* is a unique max-G decision and suppose that x^* is not a SPM. Then there exists $x^+ \in \mathbf{R}^n$, $x^* \neq x^+$, $\mu_i(x^*) \leq \mu_i(x^+)$, for all inI. Then by monotonicity of G we obtain

$$G(\mu_1(x^*), \ldots, \mu_m(x^*)) \leq G(\mu_1(x^+), \ldots, \mu_m(x^+)),$$

which is a contradiction with the uniqueness of $x^* \in X_G$. ■

The proof of the following proposition is a slight adaptation of that of Proposition 7.22 and requires that G dominates T_M, i.e., $G \gg T_M$, according to Definition 5.30 in Chapter 5.

PROPOSITION 7.30 *Let μ_i , $i \in I$, be the membership functions of fuzzy criteria on $\mathbf{R}^n$, strictly quasiconcave on their convex supports* $\mathrm{Supp}(\mu_i)$. *Let G be a strictly monotone aggregation operator such that G dominates T_M and let $x^* \in X_G$ with*

$$\min\{\mu_i(x^*) \mid i \in I\} > 0. \tag{7.13}$$

Then

$$X_G = \{x^*\},$$

i.e., x^* is a unique max-G decision.

PROOF. Let $\varphi(x) = G(\mu_1(x), \ldots, \mu_m(x))$, where $x \in \mathbf{R}^n$, and suppose that there exists $x' \in X_M$, $x^* \neq x'$. Then by (7.13)

$$\varphi(x') = \varphi(x^*) > 0. \tag{7.14}$$

By (7.13) we have also $x^*, x' \in X = \bigcap_{i \in I} \mathrm{Supp}(\mu_i)$, where X is convex. Since G is a strictly monotone aggregation operator and μ_i are strictly quasiconcave, it follows that φ is strictly quasiconcave on X. Then we obtain for $x^+ = \lambda x' + (1-\lambda)x^*$ and $0 < \lambda < 1$:

$$\begin{aligned} \varphi(x^+) &= G(\mu_1(\lambda x' + (1-\lambda)x^*), \ldots, \mu_m(\lambda x' + (1-\lambda)x^*)) \\ &> G(\min\{\mu_1(x'), \mu_1(x^*)\}, \ldots, \min\{\mu_m(x'), \mu_m(x^*)\}). \end{aligned}$$

As G dominates T_M, we obtain

$$\begin{aligned} &G(\min\{\mu_1(x'), \mu_1(x^*)\}, \ldots, \min\{\mu_m(x'), \mu_m(x^*)\}) \\ &\qquad \geq \min\{G(\mu_1(x'), \ldots, \mu_m(x')), G(\mu_1(x^*), \ldots, \mu_m(x^*))\}. \end{aligned}$$

Combining the last two inequalities with (7.14), we obtain

$$\varphi(x^+) > \min\{\varphi(x'), \varphi(x^*)\} = \varphi(x^*),$$

which contradicts the fact that $\varphi(x^*) = \max\{\varphi(x) \mid x \in X\}$. Consequently, X_G consists of only one element. ■

For fuzzy criteria μ_i, $i \in I$, an aggregation operator G and $x^* \in X_G$ satisfying the assumptions of Proposition 7.30, it follows that $x^* \in X_{SP}$.

6. Aggregation of Fuzzy Criteria

In this section we investigate the problem of aggregation of several fuzzy criteria μ_i on $\mathbf{R}^n$ with additional properties related to generalized quasiconcavity. We present sufficient conditions which secure some attractive properties. The proofs of the following propositions are omitted as they can be obtained by slight modifications of the corresponding Propositions 5.25, 5.27 and 5.31 in Chapter 5.

PROPOSITION 7.31 *Let $\mu_i : \mathbf{R}^n \to [0,1], i \in I$, be T_D-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$ such that*

$$\bigcap_{i\in I} \mathrm{Core}(\mu_i) \neq \emptyset. \tag{7.15}$$

Let $A : [0,1]^m \to [0,1]$ be an aggregation operator. Then $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by

$$\psi(x) = A(\mu_1(x), \ldots, \mu_m(x)) \tag{7.16}$$

is upper-starshaped on $\mathbf{R}^n$.

PROPOSITION 7.32 *Let $\mu_i : \mathbf{R}^n \to [0,1], i \in I$, be T_D-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$ such that*

$$\mathrm{Core}(\mu_1) = \cdots = \mathrm{Core}(\mu_m) \neq \emptyset. \tag{7.17}$$

Let $A : [0,1]^m \to [0,1]$ be a strictly monotone aggregation operator. Then $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by (7.16) is T_D-quasiconcave on $\mathbf{R}^n$.

Conditions (7.15) and (7.17) are essential for the validity of the statements of Propositions 7.32 and 7.32, which has been demonstrated by means of Examples 5.26 and 5.28.

PROPOSITION 7.33 *Let T be a t-norm, $\mu_i : \mathbf{R}^n \to [0,1], i \in I$, be T-quasiconcave membership functions of fuzzy criteria. Let A be an aggregation operator, and let A dominate T. Then $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by (7.16) is T-quasiconcave on $\mathbf{R}^n$.*

As it was already mentioned in Chapter 4, every t-norm T dominates T (reflexivity) and the minimum t-norm T_M dominates any other t-norm T. Accordingly, we obtain the following consequence of Proposition 7.33.

COROLLARY 7.34 *Let T be a t-norm, $\mu_i : \mathbf{R}^n \to [0,1], i \in I$, be T-quasiconcave membership functions of fuzzy criteria. Then $\varphi_j : \mathbf{R}^n \to [0,1]$, $j = 1, 2$, defined by*

$$\varphi_1(x) = T(\mu_1(x), \ldots, \mu_m(x)), \quad x \in \mathbf{R}^n, \tag{7.18}$$

$$\varphi_2(x) = T_M(\mu_1(x), \ldots, \mu_m(x)), \quad x \in \mathbf{R}^n, \tag{7.19}$$

are also T-quasiconcave on $\mathbf{R}^n$.

7. Extremal Properties

In this section we derive several results concerning relations between local and global maximizers (i.e. max-A decisions) of some aggregations of fuzzy criteria. For this purpose we apply the local-global properties of generalized concave functions from Chapter 3, Theorems 3.17 and 3.18, in a combination with the results on aggregation operators in Chapter 5, Propositions 5.25 and 5.31.

THEOREM 7.35 *Let $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be T-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$ such that*

$$\bigcap_{i\in I} \mathrm{Core}(\mu_i) \neq \emptyset.$$

Let $A : [0,1]^m \to [0,1]$ be an aggregation operator. If $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by

$$\psi(x) = A(\mu_1(x), \ldots, \mu_m(x)) \tag{7.20}$$

attains its strict local maximizer at $\bar{x} \in \mathbf{R}^n$, then $\bar{x}$ is a strict global maximizer of ψ on $\mathbf{R}^n$.

PROOF. By Proposition 7.31, ψ is upper-starshaped on $\mathbf{R}^n$. Now, by Theorem 3.17, we obtain the required result. ■

THEOREM 7.36 *Let T be a t-norm, $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be T-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$. Let A be an aggregation operator and let A dominate T. If $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by (7.20) attains its strict local maximum at $\bar{x} \in \mathbf{R}^n$, then $\bar{x}$ is a strict global maximizer of ψ on $\mathbf{R}^n$.*

PROOF. By Proposition 5.31, function ψ defined by (7.20) is T-quasiconcave on $\mathbf{R}^n$. Since each T-quasiconcave function on $\mathbf{R}^n$ is upper-quasiconnected on $\mathbf{R}^n$, the statement follows from Theorem 3.17. ■

THEOREM 7.37 *Let T be a t-norm, let $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be semistrictly T-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$. Let A be a strictly monotone aggregation operator and let A dominate T. If $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by (7.20) attains its local maximizer at $\bar{x} \in \mathbf{R}^n$, then $\bar{x}$ is a global maximizer of ψ on $\mathbf{R}^n$.*

PROOF. Again, by Proposition 5.31, function ψ is T-quasiconcave on $\mathbf{R}^n$. Since each T-quasiconcave function on $\mathbf{R}^n$ is upper-quasiconnected on $\mathbf{R}^n$, the statement follows from Theorem 3.18. ■

Since each t-norm T dominates T and the minimum t-norm T_M dominates any other t-norm T, we obtain the following results.

COROLLARY 7.38 *Let T be a t-norm, $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be T-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$. If $\varphi_1 : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by (7.18) attains its strict local maximum at $\bar{x}_1 \in \mathbf{R}^n$, then $\bar{x}_1$ is a strict global maximizer of φ_1 on $\mathbf{R}^n$.*

COROLLARY 7.39 *Let T be a strict t-norm, $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be semistrictly T-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$. If $\varphi_1 : \mathbf{R}^n \to [0,1]$, defined for $x \in \mathbf{R}^n$ by*

$$\varphi_1(x) = T(\mu_1(x), \ldots, \mu_m(x)),$$

attains its local maximum at $\bar{x}_1 \in \mathbf{R}^n$, then $\bar{x}_1$ is a global maximizer of φ_1 on $\mathbf{R}^n$.

COROLLARY 7.40 *Let T be a t-norm, $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be T-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$. If $\varphi_2 : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by*

$$\varphi_2(x) = T_M(\mu_1(x), \ldots, \mu_m(x)),$$

attains its strict local maximum at $\bar{x}_2 \in \mathbf{R}^n$, then $\bar{x}_2$ is a strict global maximizer of φ_2 on $\mathbf{R}^n$.

8. Application to Location Problem

A classical problem in location theory consists in location p suppliers to cover given demands of q consumers in such a way that total shipping costs are minimized; see e.g. [27]. Consider the following mathematical model:

Let $I = \{1, 2, \ldots, q\}$ be a set of q consumers located on the plane $\mathbf{R}^2$ in the points $\mathbf{c}_i$ with coordinates $\mathbf{c}_i = (u_i, v_i) \in \mathbf{R}^2$, $i \in I$. Each consumer $i \in I$ is characterized by a given nonnegative demand b_i - an amount of products, goods, services, etc. The demands of consumers are to be satisfied by a given set of p suppliers denoted by $J = \{1, 2, \ldots, p\}$. The distance of consumer $i \in I$ located at (u_i, v_i) and supplier $j \in J$ at (x_j, y_j) is denoted by $d_{ij}(x_j, y_j)$ and defined by

$$d_{ij}(x_j, y_j) = d((u_i, v_i), (x_j, y_j)),$$

where d is an appropriate distance function (e.g. Euclidean distance). The shipping cost between i and j depends on the location of consumer i, on the distance $d_{ij}(x_j, y_j)$ of consumer i to supplier j, and, on the amount of goods z_{ij} transported from j to i. These characteristics can be expressed by the value $f_i(d_{ij}(x_j, y_j), z_{ij})$ of a cost function $f_i : [0, +\infty) \times [0, +\infty) \to \mathbf{R}^1$, $i \in I$, that is nondecreasing in both variables. The total cost of the shipment of goods

from all suppliers to all consumers is defined as the sum of the individual cost functions

$$f((x_1, y_1), \ldots, (x_p, y_p), z_{11}, \ldots, z_{qp}) = \sum_{i=1}^{q} \sum_{j=1}^{p} f_i(d_{ij}(x_j, y_j), z_{ij}).$$

The problem is to find locations of suppliers (x_j, y_j) and transported amounts z_{ij} for all consumers and suppliers such that the requirements of the consumers are covered and total shipping cost is minimal. The mathematical model of the above location problem can be formulated as follows:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{q} \sum_{j=1}^{p} f_i(d_{ij}(x_j, y_j), z_{ij}) \\ \text{subject to} & \sum_{j=1}^{p} z_{ij} \geq b_i, \quad i \in I,\ j \in J, \\ & z_{ij} \geq 0, \ \ (x_j, y_j) \in \mathbf{R}^2. \end{array} \tag{7.21}$$

Problem (7.21) is a constrained nonlinear optimization problem with $2p+pq$ variables x_j, y_j, z_{ij}. We assume that functions f_i have the following simple form:

$$f_i(d_{ij}(x_j, y_j), z_{ij}) = \alpha_i \cdot d_{ij}(x_j, y_j) \cdot z_{ij},$$

where $\alpha_i > 0$ are constant coefficients, and distance function d_{ij} is defined as

$$d_{ij}(x_j, y_j) = ((u_i - x_j)^{\beta} + (v_i - y_j)^{\beta})^{1/\beta},$$

where $i \in I$, $j \in J$ and $\beta > 0$. Problem (7.21), even in the above simple form, is difficult to solve numerically because of its nonlinearities which bring numerous local optima.

In order to transform problem (7.21) in a more tractable form, we consider the objective function as a utility or satisfaction function μ, such that μ maps the Cartesian product $\mathbf{R}^{2p} \times \mathbf{R}^q$ into $[0, 1]$. Then

$$\mu((x_1, y_1), \ldots, (x_p, y_p), b_1, \ldots, b_q) = 0$$

denotes the total dissatisfaction (zero utility) with location $(x_j, y_j) \in \mathbf{R}^2, j \in J$, and supplied amounts b_i, $i \in I$. On the other hand,

$$\mu((x_1, y_1), \ldots, (x_p, y_p), b_1, \ldots, b_q) = 1$$

denotes the maximal total satisfaction (or, maximal utility) with location $(x_j, y_j) \in \mathbf{R}^2$, $j \in J$ and supplied amounts.

Depending on the required amount b_i, an individual consumer $i \in I$ may express his satisfaction with the supplier $j \in J$ located at (x_j, y_j) by membership grade $\mu_{ij}(x_j, y_j, b_i)$, where membership function $\mu_{ij} : \mathbf{R}^2 \times \mathbf{R}^1 \to [0, 1]$ satisfies condition $\mu_{ij}(u_i, v_i, b_i) = 1$, i.e., the maximal satisfaction is equal to

1, provided that the facility $j \in J$ is located at the same place as the consumer $i \in I$.

The individual satisfaction expressed by the function $\mu_i : \mathbf{R}^{2p} \times \mathbf{R}^1 \to [0,1]$ of the consumer $i \in I$ with the amount b_i, and with suppliers located at $(x_1, y_1), (x_2, y_2), \ldots, (x_p, y_p)$, is defined by the satisfaction of the location of the facility with maximal value, that is,

$$\mu_i((x_1, y_1), \ldots, (x_p, y_p), b_i) = \max\{\mu_{i1}(x_1, y_1, b_i), \ldots, \mu_{ip}(x_p, y_p, b_i)\}.$$

More generally, we can apply a compensative aggregation operator with the values inbetween max and min. The supplier $j \in J$ with the maximal grade of satisfaction $\mu_{ij}(x_j, y_j, b_i)$ will cover the required amount b_i. If there are more such suppliers, then they share amount b_i equally. The above formula can be generalized by using an aggregation operator A for all $i \in I$ as follows

$$\mu_i((x_1, y_1), \ldots, (x_p, y_p), b_i) = A(\mu_{i1}(x_1, y_1, b_i), \ldots, \mu_{ip}(x_p, y_p, b_i)).$$

Moving the location point of a supplier along the path from a location (x, y) toward the location site of consumer i at $\mathbf{c}_i = (u_i, v_i)$, it is natural to assume that satisfaction grade of consumer i is increasing, or, at least non-decreasing, provided that b_i is constant. This assumption results in (Φ, Ψ)-concavity (e.g. T-quasiconcavity) requirement of membership functions μ_{ij} on $\mathbf{R}^2$.

On the other hand, with the given location (x_j, y_j) of supplier j, satisfaction grade $\mu_{ij}(x_j, y_j, b)$ is nonincreasing in variable b.

We obtain $\mathbf{c}_i = (u_i, v_i) \in \bigcap_{j \in J} \mathrm{Core}(\mu_{ij})$. If all μ_{ij} are T-quasiconcave on $\mathbf{R}^2$, then by Proposition 7.31, individual satisfaction μ_i of consumer i is upper starshaped on $\mathbf{R}^{2p}$.

The total satisfaction with (or, utility of) locations $(x_j, y_j) \in \mathbf{R}^2$, $j \in J$, and required amounts b_i, $i \in I$, is defined as an aggregation of the individual satisfaction grades, e.g., a minimal satisfaction, or, more generally, the value of an aggregation operator G, e.g., a t-norm T, as follows:

$$\begin{aligned} &\mu((x_1, y_1), \ldots, (x_p, y_p), b_1, \ldots, b_q) \\ &\quad = G(A(\mu_{11}(x_1, y_1, b_1), \ldots, \mu_{1p}(x_p, y_p, b_1)), \\ &\qquad \ldots, A(\mu_{q1}(x_1, y_1, b_q), \ldots, \mu_{qp}(x_p, y_p, b_q))). \end{aligned} \tag{7.22}$$

Instead of considering the constrained optimization problem (7.21), we now consider the following unconstrained problem of optimal location:

$$\text{maximize the value of } \mu \text{ defined by (7.22) over } \mathbf{R}^2. \tag{7.23}$$

As a problem of unconstrained optimization, (7.23) can be numerically more easily tractable than the original problem (7.21). Notice also that problem (7.23) has only $2p$ variables, whereas problem (7.21) has $2p+pq$ variables.

We illustrate the above approach in the following two numerical examples.

EXAMPLE 7.41 Consider the location problem with three consumers and one supplier given by $(u_1, v_1) = (0,1)$, $(u_2, v_2) = (0,2)$, $(u_3, v_3) = (3,0)$, $b_1 = 10$, $b_2 = 20$, $b_3 = 30$.

First, let us deal with the classical problem (7.21) with $\alpha_i = 1$, $i = 1,2,3$, $\beta = 2$. Then the problem becomes that of minimizing:

$$f(x, y, z_1, z_2, z_3)$$
$$= z_1(x^2 + (1-y)^2)^{1/2} + z_2(x^2 + (2-y)^2)^{1/2} + z_3((x-3)^2 + y^2)^{1/2},$$

subject to $z_1 \geq 10$, $z_2 \geq 20$, $z_3 \geq 30$, and $(x, y) \in \mathbf{R}^2$.

The optimal location of the supplier has been found to be

$$(x^C, y^C, z_1^C, z_2^C, z_3^C) = (3, 0, 10, 20, 30),$$

with the minimum cost $f(x^C, y^C, z_1^C, z_2^C, z_3^C) = 103.73$.

Applying the alternative approach, we assume that the individual satisfaction of consumer i, located at $\mathbf{c}_i = (u_i, v_i)$ with location of the supplier at (x, y) and with demand b_i, is given by the following membership (satisfaction) function:

$$\mu_i(x, y, b_i) = \frac{1}{1 + b_i((x-u_i)^2 + (y-v_i)^2)^{1/2}}.$$

We shall investigate the problem with two different aggregation operators, particularly, t-norms.

If we consider the minimum t-norm, i.e.,

$$G(u, v) = T_M(u, v) = \min\{u, v\},$$

then, for

$$\begin{aligned}
\mu_1(x, y, 10) &= \frac{1}{1 + 10(x^2 + (y-1)^2)^{1/2}}, \\
\mu_2(x, y, 20) &= \frac{1}{1 + 20(x^2 + (y-2)^2)^{1/2}}, \\
\mu_3(x, y, 30) &= \frac{1}{1 + 30((x-3)^2 + y^2)^{1/2}},
\end{aligned}$$

we solve the maximum satisfaction problem: Maximize

$$\mu_M(x, y, 10, 20, 30) = \min\{\mu_1(x, y, 10), \mu_2(x, y, 20), \mu_3(x, y, 30)\}$$

subject to $(x, y) \in \mathbf{R}^2$.

The optimal location of the supplier has been found to be

$$(x^M, y^M) = (1.8, 0.8)$$

with the optimal membership function value

$$\mu_M(1.8, 0.8, 10, 20, 30) = 0.023,$$

see Figure 7.4.

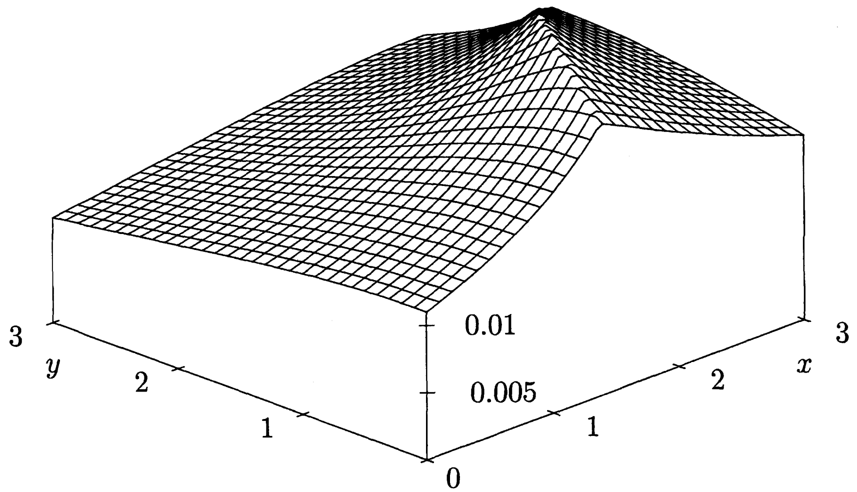

Figure 7.4.

Comparing with the optimal solution of the classical problem, we obtain the membership function value

$$\mu_M(x^C, y^C, 10, 20, 30) = 0.013.$$

On the other hand, the cost is $f(x^M, y^M, 10, 20, 30) = 104.64$.

Now, as an aggregation operator we consider the product t-norm, i.e.,

$$G(u, v) = T_P(u, v) = u \cdot v.$$

We solve the maximum satisfaction problem: Maximize

$$\mu_P(x, y, 10, 20, 30) = \mu_1(x, y, 10) \cdot \mu_2(x, y, 20) \cdot \mu_3(x, y, 30),$$

subject to $(x, y) \in \mathbf{R}^2$.

The optimal location of the supplier has been found to be $(x^P, y^P) = (0, 1)$ with the optimal membership function value $\mu_P(0, 1, 10, 20, 30) = 0.0005$, see Figure 7.5. Moreover, we get

$$\mu_p(x^C, y^C, 10, 20, 30) = 0.0004$$

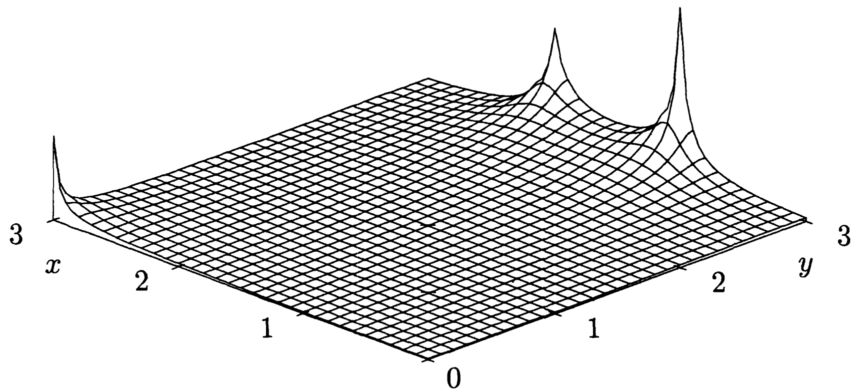

Figure 7.5.

and

$$\mu_P(x^M, y^M, 10, 20, 30) = 0.00002.$$

The cost of this solution is $f\left(x^P, y^P, 10, 20, 30\right) = 114.87$. The obtained results are summarized in the following table.

	x	y	f	μ_M	μ_P
1.	3.0	0.0	103.7	0.01	0.00041
2.	1.8	0.8	104.6	0.02	0.00002
3.	0.0	1.0	114.9	0.01	0.00050

In the above table we can see differences between the results of the individual approaches. In Row 1., the results of solving the classical problem (7.21) are displayed. In Row 2., the solution of maximum satisfaction problem with the aggregation operators being minimum t-norm and maximum t-conorm is presented. In Row 3., the results of the same problem with the aggregation operators being product t-norm and t-conorm are given. Depending both on input information and decision-making requirements, various locations of the supplier may be optimal. □

EXAMPLE 7.42 Consider the same problem as in Example 7.41 with but now with two suppliers to be optimally located.

Again, we begin with the classical problem (7.21) with $a_i = 1, i = 1, 2, 3$, $\beta = 2$. Then the problem to solve is to minimize the cost function

$$\begin{aligned} f(x_1, y_1, x_2, y_2, z_{11}, z_{12}, z_{21}, z_{22}, z_{31}, z_{32}) \\ = \quad & z_{11}(x_1^2 + (1 - y_1)^2)^{1/2} + z_{12}(x_2^2 + (1 - y_2)^2)^{1/2} \\ & + z_{21}(x_1^2 + (2 - y_1)^2)^{1/2} + z_{22}(x_2^2 + (2 - y_2)^2)^{1/2} \\ & + z_{31}((x_1 - 3)^2 + y_1^2)^{1/2} + z_{32}((x_2 - 3)^2 + y_2^2)^{1/2}, \end{aligned} \tag{7.24}$$

subject to

$$\begin{gathered} z_{11} + z_{12} \geq 10, \ z_{21} + z_{22} \geq 20, \ z_{31} + z_{32} \geq 30, \\ z_{ij} \geq 0, \ (x_j, y_j) \in \mathbf{R}^2, \ i = 1, 2, 3, \ j = 1, 2. \end{gathered}$$

The optimal solution, i.e., the locations of the facilities and shipment amounts, have been found to be

$$(x_1^C, y_1^C, x_2^C, y_2^C, z_{12}^C, z_{21}^C, z_{22}^C, z_{31}^C, z_{32}^C) = (0, 2, 3, 0, 10, 0, 20, 0, 0, 30),$$

whereas the minimum cost is

$$f(0, 2, 3, 0, 10, 0, 20, 0, 0, 30) = 10.$$

Applying our approach, the individual satisfaction of consumer $i \in I$ located at (u_i, v_i) with location of the facility at (x_j, y_j), $j \in J = \{1, 2\}$, and with demand b_i, is given by the membership (satisfaction) function

$$\mu_{ij}(x_j, y_j, b_i) = \frac{1}{1 + b_i((x_j - u_i)^2 + (y_j - v_i)^2)^{1/2}}. \tag{7.25}$$

We investigate the problem again with two different aggregation operators, particularly, t-norms and t-conorms.

First, the aggregation operator $S_M = \max$ is used for aggregating the suppliers, $T_M = \min$ is used for combining consumers. According to (7.23) we solve the optimization problem: Maximize

$$\mu_M((x_1, y_1), (x_2, y_2), 10, 20, 30) = \min \left\{ \begin{array}{l} \max\limits_{j \in J}\{\mu_{1j}(x_j, y_j, 10)\} \\ \max\limits_{j \in J}\{\mu_{2j}(x_j, y_j, 20)\} \\ \max\limits_{j \in J}\{\mu_{3j}(x_j, y_j, 30)\} \end{array} \right\},$$

subject to $(x_1, y_1, x_2, y_2) \in \mathbf{R}^4$.

The optimal locations of the facilities has been computed as

$$(x_1^M, y_1^M, x_2^M, y_2^M) = (3, 0, 0, 5/3)$$

with the optimal membership value

$$\mu_M((3,0),(0,5/3),10,20,30) = 0.130.$$

Notice that membership functions (7.25) are all T_M-quasiconcave on $\mathbf{R}^4$ and, by Proposition 7.33, μ_M is also T_M-quasiconcave on $\mathbf{R}^4$.

Second, the operator $S_P(u,v) = u + v - u \cdot v$ is used for aggregating the suppliers, $T_P(u,v) = u \cdot v$ is used for combining consumers. Again, by (7.23) we solve the maximum satisfaction problem: Maximize

$$\begin{aligned}&\mu_P((x_1,y_1),(x_2,y_2),b_1,b_2,b_3)\\ &= \textstyle\prod_{i\in I}(\mu_{i1}(x_1,y_1,b_i) + \mu_{i2}(x_2,y_2,b_i) - \mu_{i1}(x_1,y_1,b_i)\cdot\mu_{i2}(x_2,y_2,b_i)),\end{aligned}$$

subject to $(x_1,y_1,x_2,y_2) \in \mathbf{R}^4$.

The optimal locations of the facilities have been found as

$$(x_1^P, y_1^P, x_2^P, y_2^P) = (3,0,0,2)$$

with the optimal membership value

$$\mu_P((3,0),(0,2),10,20,30) = 0.119,$$

being the same as the optimal solution of classical problem (7.24). The results are summarized in the following table.

	x_1	y_1	x_2	y_2	f	μ_M	μ_P
1.	3.0	0.0	0.0	2.0	10.0	0.090	0.119
2.	3.0	0.0	0.0	1.67	13.3	0.130	0.022
3.	3.0	0.0	0.0	2.0	10.0	0.090	0.119

In the above table we can see the differences between the results of the individual problems. Solving classical problem we obtain the same results as in the maximum satisfaction problem with the aggregation operators being the product t-norm T_P and t-conorm S_P. Notice again, that membership functions (7.25) are T_P-quasiconcave on $\mathbf{R}^4$ and by Proposition 7.33 μ_P is T_P-quasiconcave on $\mathbf{R}^4$. □

9. Application in Engineering Design

Innovative product development requires high quality and resource-efficient engineering design. Traditionally, it is common for engineers to evaluate promising design alternatives one by one. Such an approach does not consider the nature of imprecision of the design process and leads to expensive design computations. At the stage where technical solution concepts are being generated, the description of a design is largely vague or imprecise. The need for a

methodology to represent and manipulate imprecision is greatest in the early, preliminary phases of engineering design, where the designer is most unsure of the final dimensions and shapes, material and properties and performance of the completed design, see [2].

Because design imprecision concerns the choice of design variable values used to describe an product or process, the designer's preference is used to quantify the imprecision with which design variables are known. Preferences, modeled here by quasiconcave membership functions, denote either subjective or objective information that may be quantified and included in the evaluation of design alternatives.

Each design variable is characterized by the membership function

$$\mu_i : \mathbf{R}^{k_i} \to [0,1], \quad i \in I = \{1, 2, \ldots, n\},$$

where the value $\mu_i(x_i)$ of the membership function specifies the design preference of the design parameter x_i; its nature is possibly multi-dimensional, n is the number of variables. This preference function, which may arise objectively (e.g., cost of the parameter), or subjectively (e.g., from experience), is used to quantify the imprecision associated with the design variable. Thus the designer's experience and judgment are incorporated into the design evaluations. In practice, the design variables-preferences are divided into two groups: individual design preferences ($D = \{1, 2, \ldots, m\}$) and individual customers preferences - functional requirements ($P = \{m+1, \ldots, n\}$).

In order to evaluate a design, the various individual preferences must be combined or aggregated to give a single, overall measure. This aggregation, in practice, occurs in two stages, see [2].

First, the individual design preferences μ_i, $i \in D$, are aggregated into the combined design preference μ_D by an aggregation operator $A_D : [0,1]^m \to [0,1]$, and the customer individual preferences μ_i, $i \in P$, are aggregated by an aggregation operator $A_P : [0,1]^{n-m} \to [0,1]$ into the combined design preference μ_P, that is,

$$\begin{aligned} \mu_D(x_1, \ldots, x_m) &= A_D(\mu_1, \ldots, \mu_m)(x_1, \ldots, x_m) \\ &= A_D(\mu_1(x_1), \ldots, \mu_m(x_m)), \end{aligned}$$

and

$$\begin{aligned} \mu_P(x_{m+1}, \ldots, x_n) &= A_P(\mu_{m+1}, \ldots, \mu_n)(x_{m+1}, \ldots, x_n) \\ &= A_P(\mu_{m+1}(x_{m+1}), \ldots, \mu_n(x_n)). \end{aligned}$$

Then we combine them by an aggregation operator A_O to obtain an overall preference μ_O from the following formula:

$$\begin{aligned} \mu_O &= A_O(\mu_D, \mu_P) \\ &= A_O(A_D(\mu_1, \ldots, \mu_m), A_P(\mu_{m+1}, \ldots, \mu_n)). \end{aligned}$$

In Definition 5.1 it is required that each aggregation operator A is monotone and satisfies the boundary conditions $A(0, 0, \ldots, 0) = 0$, and $A(1, 1, \ldots, 1) = 1$. In engineering design it is often required that aggregation operators are continuous and idempotent. The last condition restricts the class of feasible aggregation operators to the operators between the t-norm T_M and t-conorm S_M.

For some preferences of the system of engineering design, for example, where the failure of one component results in the failure of the whole system, the non-compensating aggregation operators such as minimum T_M should be applied. On the other hand, a better performance of some component can compensate some worse performance of another component. In other words, a lower membership value of some design variable can be compensated by a higher value of some other variable. Such preferences can be aggregated by compensative aggregation operators, e.g., averaging operators, see Chapter 5. Notice that by Definition 5.2, the t-norm T_M is considered also as a compensative operator, however in some other sense.

The problem of engineering design is to find an optimal configuration of the design parameters, i.e., to maximize

$$A_O(A_D(\mu_1(x_1), \ldots, \mu_m(x_m)), A_P(\mu_{m+1}(x_{m+1}), \ldots, \mu_n(x_n))).$$

We illustrate this approach on a simple example.

EXAMPLE 7.43 *(Car design)* Consider 4 design variables, two of them are individual preferences:

- μ_1 - maximal speed,
- μ_2 - time to reach 100km/hour,

defined as follows, see Figure 7.6 and Figure 7.7:

$$\mu_1(x) = \begin{cases} \frac{1}{1+0.25(160-x)} & \text{if } 0 \leq x \leq 160, \\ 1 & \text{if } x > 160. \end{cases}$$

$$\mu_2(y) = \begin{cases} 1 & \text{if } 0 \leq y \leq 7, \\ \frac{1}{1+0.3(y-7)} & \text{if } y > 7. \end{cases}$$

We consider compensative aggregation operator - Product t-norm T_P:

$$\mu_D(x, y) = A_D(\mu_1(x), \mu_2(y)) = T_P(\mu_1(x), \mu_2(y)),$$

particularly,

$$\mu_D(x, y) = \begin{cases} \frac{1}{1+0.25(160-x)} \cdot \frac{1}{1+0.3(y-7)} & \text{if } 0 \leq x \leq 160 \text{ and } y \geq 7, \\ \frac{1}{1+0.25(160-x)} & \text{if } 0 \leq y \leq 7, \\ \frac{1}{1+0.3(y-7)} & \text{if } x \geq 160, \\ 1 & \text{if } x \geq 160 \text{ and } 0 \leq y \leq 7, \end{cases}$$

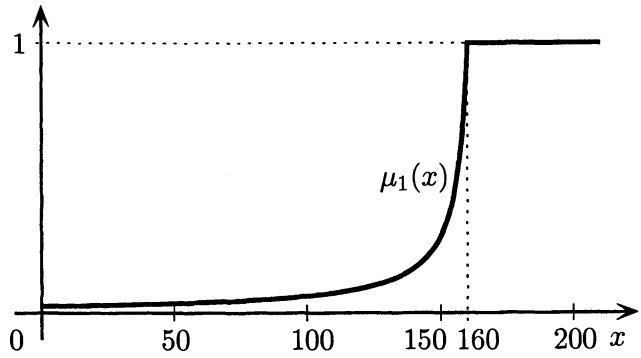

Figure 7.6.

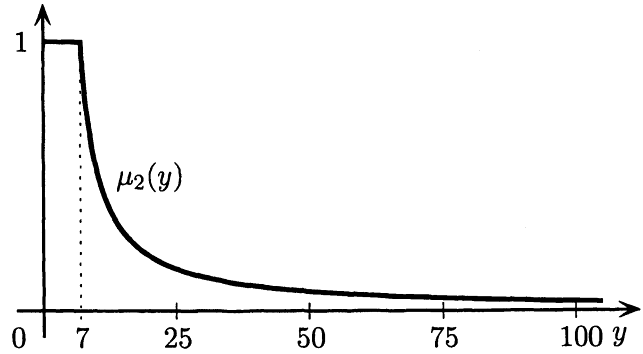

Figure 7.7.

see Figure 7.8. Notice, that μ_D is a monotone-starshaped function which is not quasiconcave on $\mathbf{R}^2_+$.

Remaining two design variables are customer preferences:

- μ_3- price of the car in \$ 1000,
- μ_4 - fuel consumption in liter/100 km.

They are defined as follows:

$$\mu_3(u) \quad = \quad \begin{cases} 1 & \text{if } 0 \leq u \leq 8, \\ \frac{1}{1+0.2(u-8)} & \text{if } u > 8. \end{cases}$$

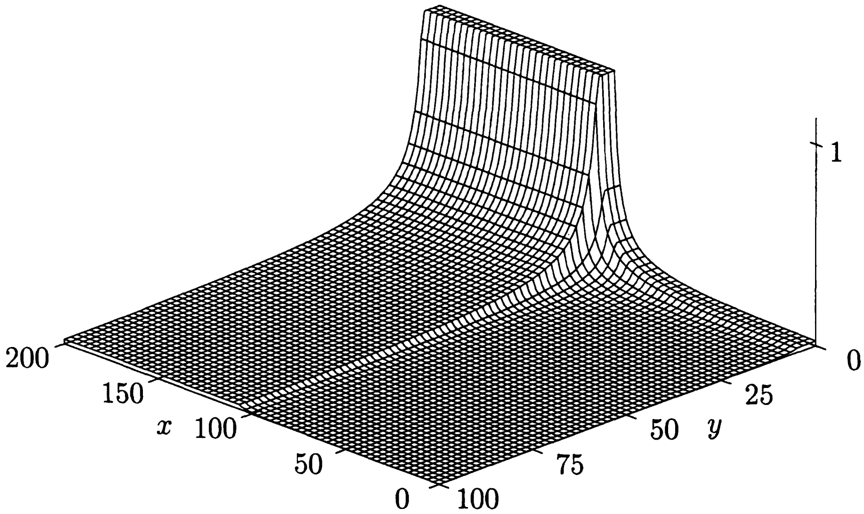

Figure 7.8.

$$\mu_4(v) \quad = \quad \begin{cases} 1 & \text{if } 0 \leq v \leq 4, \\ \frac{1}{1+0.1(v-4)} & \text{if } v > 4. \end{cases}$$

Moreover, technological and economical functional dependencies are given by the following (regression) model:

$$\begin{aligned} x &\leq \frac{400}{y} + 120, \\ u &= 0.03x - 0.3y + \frac{175}{y}, \\ v &= 0.05x - \frac{10}{y} - 2.5. \end{aligned}$$

For an aggregation operator A_P we apply a compensative operator, namely, geometric average G defined in (5.6), i.e.,

$$\mu_P(u, v) = A_P(\mu_3(u), \mu_4(v)) = G(\mu_3(u), \mu_4(v)).$$

Finally, μ_D and μ_P are combined by the aggregation operator

$$A_O = T_M,$$

to obtain the overall aggregation:

$$\mu_O(x, y, u, v) = T_M(\mu_D(x, y), \mu_P(u, v)).$$

The optimal configuration of the parameters has been found by Conjugate Gradient optimization method as follows:

$$x^* = 159.8,\ y^* = 10.1,\ u^* = 19.2,\ v^* = 6.5,\ A_O^* = 0.497.$$

$\square$

Chapter 8

FUZZY MATHEMATICAL PROGRAMMING

1. Introduction

Mathematical programming problems (MP) can be considered as decision making problems in which preferences between alternatives are described by means of objective functions on a given set of alternatives in such a way that more preferable alternatives have higher values. The values of the objective function describe effects from choices of the alternatives. In economic problems, for example, these values may reflect profits obtained when using various means of production. The set of feasible alternatives in MP problems is described implicitly by means of constraints - equations or inequalities, or both - representing relevant relationships between alternatives. In any case the results of the analysis using given formulation of the MP problem depend largely upon how adequately various factors of the real system are reflected in the description of the objective function(s) and of the constraints.

Descriptions of the objective function and of the constraints in a MP problem usually include some parameters. For example, in problems of resources allocation such parameters may represent economic parameters like costs of various types of production, labor costs requirements, shipment costs, etc. The nature of these parameters depends, of course, on the detailization accepted for the model representation, and their values are considered as data that should be exogenously used for the analysis.

Clearly, the values of such parameters depend on multiple factors not included into the formulation of the problem. Trying to make the model more representative, we often include the corresponding complex relations into it, causing that the model becomes more cumbersome and analytically unsolvable. Moreover, it can happen that such attempts to increase "the precision" of the model will be of no practical value due to the impossibility of measuring

the parameters accurately. On the other hand, the model with some fixed values of its parameters may be too crude, since these values are often chosen in a quite arbitrary way.

An intermediate approach is based on introduction into the model the means of a more adequate representation of experts understanding of the nature of the parameters in the form of fuzzy sets of their possible values. The resultant model, although not taking into account many details of the real system in question, could be a more adequate representation of the reality than that with more or less arbitrarily fixed values of the parameters. In this way we obtain a new type of MP problems containing fuzzy parameters. Treating such problems requires the application of fuzzy-set-theoretic tools in a logically consistent manner. Such treatment forms the essence of *fuzzy mathematical programming* (FMP) investigated in this chapter.

FMP and related problems have been extensively analyzed and many papers have been published displaying a variety of formulations and approaches. Most approaches to FMP problems are based on the straightforward use of the intersection of fuzzy sets representing goals and constraints and on the subsequent maximization of the resultant membership function. This approach has been mentioned by Bellman and Zadeh already in their paper [11] published in the early seventies. Later on many papers have been devoted to the problem of mathematical programming with fuzzy parameters, known under different names, mostly as fuzzy mathematical programming, but sometimes as possibilistic programming, flexible programming, vague programming, inexact programming, etc. For an extensive bibliography, see the overview paper [45].

Here we present a general approach based on a systematic extension of the traditional formulation of the MP problem. This approach is based on previous works of the authors of this book, see [84]- [99], and also on the works of many other authors, e.g., [17], [18], [20], [26], [42] - [44], [46], [47], [60], [61], [62], [63], [64], [77], [78], [108], [109], [111], [133], [138].

FMP is one of many possible approaches how to treat uncertainty in MP problems. Much research has been devoted to similarities and dissimilarities of FMP and stochastic programming (SP), see, e.g., [116] and [117]. In Chapters 8 and 9 we demonstrate that FMP (in particular, fuzzy linear programming) essentially differs from SP; FMP has its own structure and tools for investigating broad classes of optimization problems.

FMP is also different from parametric programming (PP). Problems of PP are in essence deterministic optimization problems with special variables called the parameters. The main interest in PP is focused on finding relationships between the values of parameters and optimal solutions of MP problem.

In FMP some methods and approaches motivated by SP and PP are utilized, see e.g. [18], [111]. In this book, however, algorithms and solution procedures

for MP problems are not studied, they can be found elsewhere, see e.g. the overview paper [109].

2. Modelling Reality by Fuzzy Mathematical Programming

An alternative approach to classical MP is based on introduction into the model the means of some more adequate representation of the parameters in the form of fuzzy sets. By applying FMP to real problems, one obtains a mathematical model which, although not taking into account many details of the real system in question, could be a more adequate representation of the reality than that with more or less arbitrarily chosen values of its parameters. In this way we obtain a new type of MP problems containing fuzzy parameters.

As mentioned in [109], the use of FMP models does not only avoid unrealistic modeling, but also offers a chance for reducing information costs. Then, in the first step of the interactive solution process, the fuzzy system is modeled by using only the information which the decision maker can provide without any expensive acquisition so as to obtain an initial compromise solution. Then the decision maker can perceive which further information would be required and is able to justify additional information costs.

An appropriate treatment of such problems requires proper application of special tools in a logically consistent manner. An important role in this treatment is played by *generalized concave membership functions*. Such approach forms the essence of FMP investigated in this chapter. The following treatment is based on the substance already investigated in Chapters 2, 3, 4, 5 and, particularly, Chapter 6.

First we formulate a FMP problem associated with a collection of instances of the classical MP problem. After that we define a feasible solution of FMP problem and optimal solution of FMP problem as special fuzzy sets. From practical point of view, α-cuts of these fuzzy sets are important. The main result of this chapter says that the class of all MP problems can be naturally embedded into the class of FMP problems.

3. Mathematical Programming Problems with Parameters

In the last section of Chapter 3, we consider optimization problems of the form:

$$\begin{array}{ll} \text{maximize} & f(x) \\ \text{subject to} & x \in X \end{array}$$

where f is a real-valued function on $\mathbf{R}^n$ and X is a subset of $\mathbf{R}^n$ given by means of real-valued functions $g_1, g_2, \ldots, g_m$ on $\mathbf{R}^n$ as the set of all solutions of the system

$$g_i(x) \geq 0, \quad i = 1, 2, \ldots, m.$$

Recall that f is called the objective function, g_i are called the constraint functions, the elements of X are called feasible solutions, and the global maximizers of f over X are called optimal solutions.

In this section, we assume that the set of feasible solutions is given as the set of all solutions of the system

$$\begin{aligned} g_i(x) &= b_i, \quad i = 1, 2, \ldots, m_1, \\ g_i(x) &\leq b_i, \quad i = m_1 + 1, m_1 + 2, \ldots, m_2, \\ g_i(x) &\geq b_i, \quad i = m_2 + 1, m_2 + 2, \ldots, m, \end{aligned}$$

where m_1, m_2 and m are integers such that $0 \leq m_1 \leq m_2 \leq m$, and b_i are real numbers. To simplify the notation we introduce the sets $\mathcal{M}_1 = \{1, 2, \ldots, m_1\}$, $\mathcal{M}_2 = \{m_1 + 1, m_1 + 2, \ldots, m_2\}$, $\mathcal{M}_3 = \{m_2 + 1, m_2 + 2, \ldots, m\}$ and $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$; if $m_1 = 0$ then $\mathcal{M}_1 = \emptyset$, if $m_1 = m_2$ then $\mathcal{M}_2 = \emptyset$, and if $m_2 = m$ then $\mathcal{M}_3 = \emptyset$.

For example, if f and all g_i are linear functions on $\mathbf{R}^n$ given by

$$\begin{aligned} f(x) &= c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\ g_i(x) &= a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n, \quad i \in \mathcal{M}, \end{aligned}$$

then we have the linear programming problem. If, in addition, $m_1 = 0$, $m = m_2 + n$, $b_i = 0$ for all $i \in \mathcal{M}_3$, and

$$a_{ij} = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

for all $i, j \in \mathcal{M}_3$, then we obtain the linear programming in the standard inequality form

$$\begin{aligned} \text{minimize} \quad & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\ \text{subject to} \quad & a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \leq b_i, \quad i \in \mathcal{M}_2, \\ & x_1 \geq 0,\ x_2 \geq 0, \ldots,\ x_n \geq 0. \end{aligned} \tag{8.1}$$

Notice that for given m_1, m_2, m, every instance of problem (8.1) is specified by vectors $c = (c_1, c_2, \ldots, c_n)$, $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$, $i \in \mathcal{M}_2$, from $\mathbf{R}^n$ and real numbers b_i, $i \in \mathcal{M}_2$. In practice, one often needs to solve or analyze not just one instance, but several related instances. In such a situation, certain subsets $\mathbf{C}$ and $\mathbf{P}_i$, $i \in \mathcal{M}_2$, of $\mathbf{R}^n$, and certain subsets B_i of $\mathbf{R}$ are given that specify a set of problem instances by allowing c, a_i and b_i to be arbitrary elements of the sets $\mathbf{C}$, $\mathbf{P}_i$ and B_i, respectively.

More generally, let $\mathbf{C}$ and $\mathbf{P}_i$, $i \in \mathcal{M}$, be arbitrary sets and let B_i, $i \in \mathcal{M}$, be subsets of $\mathbf{R}$. Furthermore let f and g_i, $i \in \mathcal{M}$, be real-valued functions defined on $\mathbf{R}^n \times \mathbf{C}$ and $\mathbf{R}^n \times \mathbf{P}_i$, $i \in \mathcal{M}$, respectively. Under the MP problem

with parameters we understand the collection of instances of the problem

$$\begin{aligned} \text{maximize} \quad & f(x;c) \\ \text{subject to} \quad & g_i(x;a_i) = b_i, \quad i \in \mathcal{M}_1, \\ & g_i(x;a_i) \le b_i, \quad i \in \mathcal{M}_2, \\ & g_i(x;a_i) \ge b_i, \quad i \in \mathcal{M}_3, \end{aligned} \tag{8.2}$$

obtained by all possible choices of $c \in \mathbf{C}$; $a_i \in \mathbf{P}_i$, $i \in \mathcal{M}$; $b_i \in B_i$, $i \in \mathcal{M}$. Notice that each instance is a MP problem in $\mathbf{R}^n$ with objective function $f(\cdot;c)$ and constraints given by function $g_i(\cdot;a_i)$ and numbers b_i, $i \in \mathcal{M}$. The elements of $\mathbf{C}$, $\mathbf{P}_i$ and B_i, $i \in \mathcal{M}$, will be called parameters but this should not be confused with the terminology of parametric programming,

On the one hand, in parametric programming, one also considers a set of related instances but the approach to that set of instances differs from the fuzzy approach described and applied in the next two chapters. For simplicity, consider the set of instances of linear programming problem (8.1) given as follows. All a_i and b_i are fixed and the coefficients of the objective function are affine functions of the same real variable t given by

$$c_1(t) = \alpha_1 t + \beta_1,\ c_2(t) = \alpha_2 t + \beta_2, \ldots,\ c_n(t) = \alpha_n t + \beta_n,$$

where t varies over a given interval $[t_0, t_1]$ of real numbers. The variable t is then called a parameter with values in $[t_0, t_1]$, and the main issue is to study how the optimal value and optimal solution depends on changes in values of the parameter.

On the other hand, in the fuzzy approach considered in the next two chapters, a single fuzzy optimization problem is associated with the MP problem with parameters given by (8.2).

4. Formulation of Fuzzy Mathematical Programming Problems

It seems that using the notation and concepts introduced in Chapter 6, we can associate with the MP problem with parameters given by (8.2) the fuzzy mathematical programming problem of the form

$$\begin{aligned} \text{maximize} \quad & \tilde{f}(x;\tilde{c}) \\ \text{subject to} \quad & \tilde{g}_i(x;\tilde{a}_i)\tilde{R}_i\tilde{b}_i, \quad i \in \mathcal{M}. \end{aligned} \tag{8.3}$$

However, it is not clear what do we mean by this problem. Let us clarify it.

Recall that the problem (8.2) represents a set of instances of an optimization problem in $\mathbf{R}^n$ which is given by sets of parameters $\mathbf{C}$, $\mathbf{P}_i$, B_i and functions $f : \mathbf{R}^n \times \mathbf{C} \to \mathbf{R}$, $g_i : \mathbf{R}^n \times \mathbf{P}_i \to \mathbf{R}$. For simplicity, we assume that all sets B_i, $i \in \mathcal{M}$, are equal to $\mathbf{R}$. The guiding idea of our approach is to associate

with this set of instances just one fuzzy optimization problem of the form (8.3). For this purpose, we assume that:

- $\tilde{c}$ is a fuzzy subset of $\mathbf{C}$ with membership function $\mu_{\tilde{c}} : \mathbf{C} \to [0, 1]$;
- $\tilde{a}_i$ is a fuzzy subset of $\mathbf{P}_i$ with membership function $\mu_{\tilde{a}_i} : \mathbf{P}_i \to [0, 1]$, $i \in \mathcal{M}$;
- $\tilde{b}_i$ is a fuzzy quantity with membership function $\mu_{\tilde{b}_i} : \mathbf{R} \to [0, 1], i \in \mathcal{M}$;
- $\tilde{R}_i$ is a fuzzy relation with membership function $\mu_{\tilde{R}_i} : \mathcal{F}(\mathbf{R}) \to \mathcal{F}(\mathbf{R})$, $i \in \mathcal{M}$, given by

$$\begin{aligned}
&\mu_{\tilde{R}_i}(\tilde{g}_i(x; \tilde{a}_i), \tilde{b}_i) \\
&\qquad = \sup\{T(\mu_{R_i}(u, v), T(\mu_{\tilde{g}_i(x;\tilde{a}_i)}(u), \mu_{\tilde{b}_i}(v))) \mid u, v \in \mathbf{R}\} \\
&\qquad = \sup\{T(\mu_{\tilde{g}_i(x;\tilde{a}_i)}(u), \mu_{\tilde{b}_i}(v)) \mid uR_iv\};
\end{aligned} \tag{8.4}$$

- $\tilde{f}(x; \tilde{c})$ is the value of the fuzzy extension of $f(x; \cdot)$ at $\tilde{c}$ given by

$$\mu_{\tilde{f}(x;\tilde{c})}(t) = \begin{cases} \sup\{\mu_{\tilde{c}}(c) \mid c \in \mathbf{C},\ f(x; c) = t\} & \text{if } f^{-1}(x; t) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{8.5}$$

- $g_i(x; \tilde{a}_i)$ is the value of the fuzzy extension of $g_i(x; \cdot)$ at $\tilde{a}_i$ given by

$$\mu_{\tilde{g}_i(x;\tilde{a}_i)}(t) = \begin{cases} \sup\{\mu_{\tilde{a}_i}(a) \mid a \in \mathbf{P}_i,\ g_i(x; a) = t\} & \text{if } g_i^{-1}(x; t) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Having clarified the meaning of individual terms appearing in (8.3), we must also clarify the meaning of the words "maximize" and "subject to", because it is not clear what do we mean by maximization of fuzzy set $\tilde{f}(x; \tilde{c})$ and by satisfaction of constraints $\tilde{g}_i(x, \tilde{a}_i)\tilde{R}_i\tilde{b}_i$. We need to define some counterparts of the concepts of feasible solutions and optimal solutions of MP problems.

The counterpart of the feasibility is described in detail in the next section. Regarding the optimality, we have to cope with the fact that there is no sufficiently universal and natural linear ordering of the set of fuzzy values $\tilde{f}(x; \tilde{c})$. Thus we have to define a suitable ordering on the family $\tilde{f}(x; \tilde{c})$, $x \in \mathbf{R}^n$ of fuzzy sets that allows for some reasonable "maximization". This will be done later in this chapter by means of an exogenously given fuzzy quantity $\tilde{b}_0$, called the fuzzy goal, and an additional fuzzy relation $\tilde{R}_0$. The fuzzy objective is then treated as another constraint $\tilde{f}(x; \tilde{c})\tilde{R}_0\tilde{b}_0$, and the maximization of the objective function means finding a maximizer of the membership function $\mu_{\tilde{R}_0}(\tilde{f}(x; \tilde{c}), \tilde{b}_0)$. This approach is frequently used in the literature, see the overview paper [107]. For alternative approaches, see [25],[30],[82].

5. Feasible Solutions of FMP Problems

The fuzzy relation $\tilde{R}_i$ is considered to be an extension of either the usual equality relation "$=$" or one of the inequality relations "$\leq$" and "$\geq$", see Examples 6.54, 6.55 and 6.56 in Chapter 6. Numerous authors, however, pointed out some disadvantages of T-fuzzy extensions of equality and inequality relations. Therefore, a number of special fuzzy relations for comparing left and right sides of constraints (8.3) have been proposed; see e.g. [26], [43], [60], [61], [62], [63], [64], [77], [78], [84], [97] or [109]. We shall use some extensions of the usual equality and inequality relations in the constraints of FMP (8.3), however, not necessarily T-fuzzy extensions. In the following definition, for the sake of generality of the presentation, we consider $\tilde{R}_i$ as fuzzy relations.

In the following definition, for each $i \in \mathcal{M}$, $\tilde{a}_i$ is a fuzzy subset of a given set P_i of parameters, $\tilde{b}_i$ is a fuzzy quantity, and g_i is real-valued function defined on $\mathbf{R}^n \times \mathbf{P}_i$

DEFINITION 8.1 *Let $\tilde{R}_i$, $i \in \mathcal{M}$, be fuzzy relations with the membership functions $\mu_{\tilde{R}_i} : \mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}) \to [0,1]$, and let A be an aggregation operator. The fuzzy subset $\tilde{X}$ of $\mathbf{R}^n$ whose membership function $\mu_{\tilde{X}}$ is defined by*

$$\mu_{\tilde{X}}(x) = A(\mu_{\tilde{R}_1}(\tilde{g}_1(x;\tilde{a}_1),\tilde{b}_1),\ldots,\mu_{\tilde{R}_m}(\tilde{g}_m(x;\tilde{a}_m),\tilde{b}_m)) \qquad (8.6)$$

is called the feasible solution *of the FMP problem (8.3). For $\alpha \in (0,1]$, the elements of the α-cut are called α-*feasible solutions *of (8.3), and a point $\bar{x}$ such that $\mu_{\tilde{X}}(\bar{x}) = \mathrm{Hgt}(\tilde{X})$ is called a* max-feasible solution *of (8.3).*

Notice that the feasible solution of a FMP problem is a fuzzy subset of $\mathbf{R}^n$. For $x \in \mathbf{R}^n$ the interpretation of $\mu_{\tilde{X}}(x)$ depends on the interpretation of uncertain parameters of the FMP problem. For instance, within the framework of possibility theory, the membership functions of the parameters are explained as possibility degrees and $\mu_{\tilde{X}}(x)$ denotes the possibility of the event that $x \in \mathbf{R}^n$ belongs to the feasible solution of the FMP problem. Some other interpretations have been also applied, see e.g. [20], [42] or [109].

On the other hand, each α-feasible solution is a point belonging to the α-cut of the feasible solution $\tilde{X}$ and the same holds for the max-feasible solution, which is a special α-feasible solution with $\alpha = \mathrm{Hgt}(\tilde{X})$. If a decision maker specifies the degree of feasibility $\alpha \in [0,1]$ (the degree of possibility, satisfaction etc.), then a vector $x \in \mathbf{R}^n$ with $\mu_{\tilde{X}}(x) \geq \alpha$ is an α-feasible solution of the respective FMP problem.

Considering the ith constraint of problem (8.3), for given x, $\tilde{a}_i$ and $\tilde{b}_i$, the value $\mu_{\tilde{R}_i}(\tilde{g}_i(x;\tilde{a}_i),\tilde{b}_i)$ from interval $[0,1]$ can be interpreted as the degree of satisfaction of this constraint.

For $i \in \mathcal{M}$, we use the following notation: by $\tilde{X}_i$ we denote the fuzzy set given by the membership function $\mu_{\tilde{X}_i}$, which is defined for all $x \in \mathbf{R}^n$ as

$$\mu_{\tilde{X}_i}(x) = \mu_{\tilde{R}_i}(\tilde{g}_i(x; \tilde{a}_i), \tilde{b}_i). \tag{8.7}$$

The fuzzy set $\tilde{X}_i$ is interpreted as the ith fuzzy constraint. All fuzzy constraints are aggregated into the feasible solution (8.6) by the aggregation operator A.

6. Properties of Feasible Solution

In this section we suppose that T is a t-norm, A is an aggregation operator, $\mathbf{P}_i$ are sets of parameters, $\tilde{R}_i$ are fuzzy relations, $i \in \mathcal{M}$. Below, in Theorem 8.2, the fuzzy relations $\tilde{R}_i$ are supposed to be fuzzy extensions of the usual binary relations on $\mathbf{R}$, but in other theorems and propositions which will follow, we suppose that a stronger requirement is satisfied. Namely, we suppose that $\tilde{R}_i$ are T-fuzzy extensions of relations R_i. For the sake of simplicity, we denote the relations only by the symbol $\tilde{R}_i$ and not by $\Psi^T(R_i)$ as it was originally introduced in Definition 6.26. Other fuzzy extensions of the usual binary relations on $\mathbf{R}$, defined earlier in Definition 6.26, will not be treated in this chapter. However, we shall use them again in Chapter 9.

Investigating the concept of the feasible solution (8.6) of the FMP problem (8.3), we first show that in the case of crisp parameters a_i and b_i, the feasible solution is also crisp.

THEOREM 8.2 *Let $\tilde{a}_i$ and $\tilde{b}_i$ in (8.3) be crisp parameters given by $a_i \in \mathbf{P}_i$ and $b_i \in \mathbf{R}$, $i \in \mathcal{M}$. For $i \in \mathcal{M}$, let $\tilde{R}_i$ be a fuzzy extension of relation R_i, where R_i stands for $=$ or $\leq$ or $\geq$, depending on whether $i \in \mathcal{M}_1$ or $i \in \mathcal{M}_2$ or $i \in \mathcal{M}_3$, respectively. Furthermore, let A be a t-norm.*

Then the feasible solution $\tilde{X}$ is a crisp set that coincides with the set X of feasible solutions of the instance of MP problem (8.2) given by a_i and b_i, $i \in \mathcal{M}$.

PROOF. We wish to show that

$$\mu_{\tilde{X}}(x) = \chi_X(x) \quad \text{for each } x \in \mathbf{R}^n.$$

Observe first that by extension principle (6.15) we obtain

$$\mu_{\tilde{g}_i(x;a_i)} = \chi_{g_i(x;a_i)} \quad \text{for all } i \in \mathcal{M}.$$

Next, for all $i \in \mathcal{M}$, we obtain by (6.23)

$$\mu_{\tilde{R}_i}(g_i(x; a_i), b_i) = 1. \tag{8.8}$$

Notice that $X = \{x \in \mathbf{R}^n \mid g_i(x; a_i) R_i b_i,\ i \in \mathcal{M}\}$, where we write $g_i(x; a_i) R_i b_i$ instead of (8.8). Applying the t-norm A on (8.8), we obtain

$$\mu_{\tilde{X}}(x) = A(\mu_{\tilde{R}_1}(g_1(x; a_1), b_1), \ldots, \mu_{\tilde{R}_m}(g_m(x; a_m), b_m)) = \chi_X(x),$$

which is the desired result. ■

Recall that, for two fuzzy subsets $\tilde{a}', \tilde{a}'' \in \mathcal{F}(\mathbf{R}^n)$, $\tilde{a}' \subset \tilde{a}''$ if and only if $\mu_{\tilde{a}'}(x) \leq \mu_{\tilde{a}''}(x)$ for all $x \in \mathbf{R}^n$, see Proposition 6.6. The following theorem shows some monotonicity of the feasible solution depending on the parameters of the FMP problem. In Theorem 8.2, we assumed that $\tilde{R}_i$ have been fuzzy extensions of the usual binary relations "$=$", "$\leq$" and "$\geq$". Here, we allow $\tilde{R}_i$ to be fuzzy extensions of more general valued relations.

THEOREM 8.3 *Let g_i be real-valued functions, $g_i : \mathbf{R}^n \times \mathbf{P}_i \to \mathbf{R}$, where $\mathbf{P}_i$ are sets of parameters. Let $\tilde{a}'_i$, $\tilde{b}'_i$, and $\tilde{a}''_i$, $\tilde{b}''_i$ be two collections of fuzzy parameters of the FMP problem. Let T be a t-norm, let $\tilde{R}_i = \Psi^T(R_i)$ be T-fuzzy extensions of valued relations R_i on $\mathbf{R}$, $i \in \mathcal{M}$, and A be an aggregation operator*

If $\tilde{X}'$ is the feasible solution of the FMP problem with the collection of parameters $\tilde{a}'_i$, $\tilde{b}'_i$, and $\tilde{X}''$ is the feasible solution of the FMP problem with the collection of parameters $\tilde{a}''_i$, $\tilde{b}''_i$ such that for all $i \in \mathcal{M}$

$$\tilde{a}'_i \subset \tilde{a}''_i \text{ and } \tilde{b}'_i \subset \tilde{b}''_i,$$

then

$$\tilde{X}' \subset \tilde{X}''.$$

PROOF. In order to prove $\tilde{X}' \subset \tilde{X}''$, we first show that $\tilde{g}_i(x; \tilde{a}'_i) \subset \tilde{g}_i(x; \tilde{a}''_i)$ for all $i \in \mathcal{M}$.

Indeed, by (6.15), for each $u \in \mathbf{R}$ and $i \in \mathcal{M}$,

$$\begin{aligned} \mu_{\tilde{g}_i(x;\tilde{a}'_i)}(u) &= \max\{0, \sup\{\mu_{\tilde{a}'_i}(a) \mid a \in \mathbf{P}_i,\ g_i(x; a) = u\}\} \\ &\leq \max\{0, \sup\{\mu_{\tilde{a}''_i}(a) \mid a \in \mathbf{P}_i,\ g_i(x; a) = u\}\} \\ &= \mu_{\tilde{g}_i(x;\tilde{a}''_i)}(u). \end{aligned}$$

Now, since $\tilde{b}'_i \subset \tilde{b}''_i$, it follows from the monotonicity of T-fuzzy extension $\tilde{R}_i$ of R_i that $\mu_{\tilde{R}_i}(\tilde{g}_i(x; \tilde{a}'_i), \tilde{b}'_i) \leq \mu_{\tilde{R}_i}(\tilde{g}_i(x; \tilde{a}''_i), \tilde{b}''_i)$. Then, applying monotonicity of A in (8.6), we obtain $\tilde{X}' \subset \tilde{X}''$. ■

COROLLARY 8.4 *Let $\tilde{a}_i$, $\tilde{b}_i$ be a collection of fuzzy parameters, and let $a_i \in \mathbf{P}_i$ and $b_i \in \mathbf{R}$ be a collection of crisp parameters such that for all $i \in \mathcal{M}$*

$$\mu_{\tilde{a}_i}(a_i) = \mu_{\tilde{b}_i}(b_i) = 1.$$

If the set X of all feasible solutions of MP problem (8.2) with the parameters a_i and b_i is nonempty, and $\tilde{X}$ is a feasible solution of FMP problem (8.3) with fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$, then for all $x \in X$

$$\mu_{\tilde{X}}(x) = 1. \tag{8.9}$$

PROOF. Observe that $a_i \subset \tilde{a}_i$, $b_i \subset \tilde{b}_i$ for all $i \in \mathcal{M}$. Then by Theorem 8.3 we obtain $X \subset \tilde{X}$, which is nothing else than (8.9). ∎

Corollary 8.4 says that if we "fuzzify" the parameters of the original crisp MP problem, then the feasible solution of the new FMP problem "fuzzifies" the original set of all feasible solutions such that the membership degree of any feasible solution of the MP problem is equal to 1.

So far, the parameters $\tilde{a}_i$ of the constraint functions g_i have been specified as fuzzy subsets of arbitrary sets $\mathbf{P}_i$, $i \in \mathcal{M}$. From now on, the space of parameters is supposed to be the k-dimensional Euclidean vector space $\mathbf{R}^k$, i.e., $\mathbf{P}_i = \mathbf{R}^k$ for all $i \in \mathcal{M}$, where k is a positive integer. Particularly, $\mu_{\tilde{a}_i} : \mathbf{R}^k \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$ are the membership functions of fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$. We shall also require compactness of fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$ and closedness of the valued relations R_i. For the rest of this section, we suppose that A and T are the minimum t-norms, i.e., $A = T = T_M$. As a result, we obtain some formulae for α-feasible solutions of FMP problem based on α-cuts of the parameters. Remember that fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$ are compact if $[\tilde{a}_i]_\alpha$ and $[\tilde{b}_i]_\alpha$ are compact for all $\alpha \in (0,1]$.

THEOREM 8.5 *Let g_i be continuous functions, $g_i : \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}$. Let $\tilde{a}_i$ and $\tilde{b}_i$ be compact fuzzy subsets of $\mathbf{R}^k$ and $\mathbf{R}$, respectively. Let $\tilde{R}_i = \Psi^T(R_i)$, i.e., $\tilde{R}_i$ be T-fuzzy extensions of closed valued relations R_i on $\mathbf{R}$, $i \in \mathcal{M}$. Then for all $\alpha \in (0,1]$*

$$[\tilde{X}]_\alpha = \bigcap_{i=1}^{m} [\tilde{X}_i]_\alpha, \tag{8.10}$$

and, moreover, for all $i \in \mathcal{M}$ we have

$$[\tilde{X}_i]_\alpha = \{x \in \mathbf{R}^n \mid \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) \geq \alpha\}. \tag{8.11}$$

PROOF. 1. Let $\alpha \in (0,1]$, $i \in \mathcal{M}$, $x \in [\tilde{X}_i]_\alpha$. Then by (8.7) we have

$$\mu_{\tilde{R}_i}(\tilde{g}_i(x; \tilde{a}_i), \tilde{b}_i) \geq \alpha. \tag{8.12}$$

First, we prove that (8.12) is valid if and only if

$$\mu_{\tilde{R}_i}([\tilde{g}_i(x; \tilde{a}_i)]_\alpha, [\tilde{b}_i]_\alpha) \geq \alpha. \tag{8.13}$$

By Definition 8.1 we obtain

$$\begin{aligned}&\mu_{\tilde{R}_i}(\tilde{g}_i(x;\tilde{a}_i),\tilde{b}_i)\\ &\qquad = \sup\{\min\{\mu_{R_i}(u,v),\min\{\mu_{\tilde{g}_i(x;\tilde{a}_i)}(u),\mu_{\tilde{b}_i}(v)\}\}\mid u,v\in\mathbf{R}\}.\end{aligned}$$

As $\tilde{g}_i(x;\tilde{a}_i)$ and $\tilde{b}_i$ are compact fuzzy sets and R_i is a closed valued relation, there exist $u^*, v^* \in \mathbf{R}$ such that

$$\mu_{\tilde{R}_i}(\tilde{g}_i(x;\tilde{a}_i),\tilde{b}_i) = \min\{\mu_{R_i}(u^*,v^*),\min\{\mu_{\tilde{g}_i(x;\tilde{a}_i)}(u^*),\mu_{\tilde{b}_i}(v^*)\}\} \geq \alpha.$$

Hence,

$$\mu_{R_i}(u^*,v^*) \geq \alpha,\ \mu_{\tilde{g}_i(x;\tilde{a}_i)}(u^*) \geq \alpha,\ \mu_{\tilde{b}_i}(v^*) \geq \alpha. \tag{8.14}$$

On the other hand, by definition

$$\begin{aligned}&\mu_{\tilde{R}_i}([\tilde{g}_i(x;\tilde{a}_i)]_\alpha,[\tilde{b}_i]_\alpha)\\ &\qquad = \sup\{\min\{\mu_{R_i}(u,v),\min\{\chi_{[\tilde{g}_i(x;\tilde{a}_i)]_\alpha}(u),\chi_{[\tilde{b}_i]_\alpha}(v)\}\}\mid u,v\in\mathbf{R}\}\\ &\qquad = \sup\{\mu_{R_i}(u,v)\mid u\in[\tilde{g}_i(x;\tilde{a}_i)]_\alpha,\ v\in[\tilde{b}_i]_\alpha\}.\end{aligned}$$

Therefore, by (8.14) we obtain

$$\mu_{\tilde{R}_i}([\tilde{g}_i(x;\tilde{a}_i)]_\alpha,[\tilde{b}_i]_\alpha) \geq \alpha.$$

The converse implication can be proved analogously. Hence, (8.12) is equivalent to (8.13).

Now, by Proposition 6.46 and Theorem 6.50, it follows that

$$[\tilde{g}_i(x;\tilde{a}_i)]_\alpha = g_i(x;[\tilde{a}_i]_\alpha). \tag{8.15}$$

Substituting (8.15) into (8.13), we obtain

$$\mu_{\tilde{R}_i}(g_i(x;[\tilde{a}_i]_\alpha),[\tilde{b}_i]_\alpha) \geq \alpha,$$

which establishes the validity of (8.11).

2. To prove (8.10), observe first that with $A = \min$ in (8.6), we have

$$\mu_{\tilde{X}}(x) = \min\{\mu_{\tilde{R}_1}(\tilde{g}_1(x;\tilde{a}_1),\tilde{b}_1),\ldots,\mu_{\tilde{R}_m}(\tilde{g}_m(x;\tilde{a}_m),\tilde{b}_m)\}. \tag{8.16}$$

Let $x \in [\tilde{X}]_\alpha$, that is $\mu_{\tilde{X}}(x) \geq \alpha$. By (8.16) this is equivalent to

$$\mu_{\tilde{R}_i}(\tilde{g}_i(x;\tilde{a}_i),\tilde{b}_i) \geq \alpha$$

for all $i \in \mathcal{M}$. Using the arguments of the first part of the proof, the last inequality is equivalent to $x \in [\tilde{X}_i]_\alpha$ for all $i \in \mathcal{M}$. In other words, $x \in \bigcap_{i=1}^m [\tilde{X}_i]_\alpha$. ■

Theorem 8.5 has some important computational aspects. Assume that in a FMP problem, we can specify a possibility (satisfaction) level $\alpha \in (0,1]$ and determine the α-cuts $[\tilde{a}_i]_\alpha$ and $[\tilde{b}_i]_\alpha$ of the fuzzy parameters. Then the formulae (8.10) and (8.11) will allow us to compute all α-feasible solutions of FMP problem without performing special computations of functions $\tilde{g}_i$.

If the valued relations R_i are binary relations similar to those in Theorem 8.2, then the statement of Theorem 8.5 can be strengthened as follows.

THEOREM 8.6 *Let g_i be continuous functions, $g_i : \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}$. Let $\tilde{a}_i$ and $\tilde{b}_i$ be compact fuzzy subsets of $\mathbf{R}^k$ and $\mathbf{R}$, respectively. For $i \in \mathcal{M}_1$, let $\tilde{R}_i = \Psi^T(=)$, i.e., let $\tilde{R}_i$ be a T-fuzzy extension of the equality relation "$=$"; for $i \in \mathcal{M}_2$, let $\tilde{R}_i = \Psi^T(\leq)$ be a T-fuzzy extension of the inequality relation "$\leq$"; and for $i \in \mathcal{M}_3$, let $\tilde{R}_i = \Psi^T(\geq)$ be a T-fuzzy extension of the inequality relation "$\geq$".*

Then for all $\alpha \in (0,1]$

$$[\tilde{X}]_\alpha = \bigcap_{i=1}^{m} [\tilde{X}_i]_\alpha,$$

and, moreover, for all $i \in \mathcal{M}$ we have

$$[\tilde{X}_i]_\alpha = \{x \in \mathbf{R}^n \mid \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1\}.$$

PROOF. 1. Let $\alpha \in (0,1]$, $i \in \mathcal{M}$, $x \in [\tilde{X}_i]_\alpha$. Then by (8.4) and (8.7) we have

$$\mu_{\tilde{R}_i}(\tilde{g}_i(x; \tilde{a}_i), \tilde{b}_i) \geq \alpha. \tag{8.17}$$

We prove first that (8.17) is valid if and only if

$$\mu_{\tilde{R}_i}([\tilde{g}_i(x; \tilde{a}_i)]_\alpha, [\tilde{b}_i]_\alpha) = 1. \tag{8.18}$$

In order to apply Corollary 6.62, for each $x \in \mathbf{R}^n$ and each $\alpha \in (0,1]$, $[\tilde{g}_i(x; \tilde{a}_i)]_\alpha$ should be compact. Indeed, this is true by Proposition 6.52. Then by Corollary 6.62, we obtain the equivalence between (8.17) and (8.18). Moreover, by Proposition 6.46 and Theorem 6.50, we have

$$[\tilde{g}_i(x; \tilde{a}_i)]_\alpha = g_i(x; [\tilde{a}_i]_\alpha). \tag{8.19}$$

Substituting (8.19) into (8.18), we obtain $\mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1$, as required.

To prove the remaining part, we can repeat the arguments of the corresponding part of the proof of Theorem 8.5. ■

Now, we shall see how the concept of generalized concavity introduced in Chapter 3 is utilized in the FMP problem. Particularly, we show that all α-feasible solutions (8.10), (8.11) are solutions of the system of inequalities on

condition that the membership functions of fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$ are upper-quasiconnected for all $i \in \mathcal{M}$.

For given $\alpha \in (0, 1]$, $i \in \mathcal{M}$, we introduce the following notation

$$\underline{G}_i(x;\alpha) = \inf\{g_i(x;a) \mid a \in [\tilde{a}_i]_\alpha\}, \tag{8.20}$$

$$\overline{G}_i(x;\alpha) = \sup\{g_i(x;a) \mid a \in [\tilde{a}_i]_\alpha\}, \tag{8.21}$$

$$\underline{b}_i(\alpha) = \inf\{b \in \mathbf{R} \mid b \in [\tilde{b}_i]_\alpha\}, \tag{8.22}$$

$$\overline{b}_i(\alpha) = \sup\{b \in \mathbf{R} \mid b \in [\tilde{b}_i]_\alpha\}. \tag{8.23}$$

THEOREM 8.7 *Let all assumptions of Theorem 8.6 be satisfied. Moreover, let the membership functions of fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$ be upper-quasiconnected for all $i \in \mathcal{M}$.*

Then for all $\alpha \in (0, 1]$, we have $x \in [\tilde{X}]_\alpha$ if and only if

$$\underline{G}_i(x;\alpha) \leq \overline{b}_i(\alpha), \quad i \in \mathcal{M}_1 \cup \mathcal{M}_2,$$

$$\overline{G}_i(x;\alpha) \geq \underline{b}_i(\alpha), \quad i \in \mathcal{M}_1 \cup \mathcal{M}_3,$$

PROOF. Let $x \in [\tilde{X}]_\alpha$. By Theorem 8.6, this is equivalent to

$$\mu_{\tilde{R}_i}(g_i(x;[\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1,$$

for all $i \in \mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$. Moreover, it follows from Proposition 6.46 and Theorem 6.50 that

$$[\tilde{g}_i(x;\tilde{a}_i)]_\alpha = g_i(x;[\tilde{a}_i]_\alpha).$$

Since the membership functions of fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$, $i \in \mathcal{M}$, are upper-quasiconnected, by Propositions 6.48 and 6.52, $[\tilde{g}_i(x;\tilde{a}_i)]_\alpha$ is closed and convex, i.e., it is a closed interval in $\mathbf{R}$. The rest of the proof follows from (8.20) - (8.23) and Theorem 8.3. ∎

If we assume that the functions g_i satisfy some convexity and concavity requirements, then we can prove that the membership function $\mu_{\tilde{X}}$ of the feasible solution $\tilde{X}$ is quasiconcave, or in other words, that $\tilde{X}$ is convex.

THEOREM 8.8 *Let all assumptions of Theorem 8.6 be satisfied. Moreover, let g_i be quasiconvex on $\mathbf{R}^n \times \mathbf{R}^k$ for $i \in \mathcal{M}_1 \cup \mathcal{M}_2$, and g_i be quasiconcave on $\mathbf{R}^n \times \mathbf{R}^k$ for $i \in \mathcal{M}_1 \cup \mathcal{M}_3$.*

Then for all $i \in \mathcal{M}$, $\tilde{X}_i$ are convex and therefore the feasible solution $\tilde{X}$ of FMP problem (8.3) is also convex.

PROOF. 1. Let $i \in \mathcal{M}_1 \cup \mathcal{M}_2$, $\alpha \in (0, 1]$. We show that $[\tilde{X}_i]_\alpha$ is convex.

Let $x_1, x_2 \in [\tilde{X}_i]_\alpha$, $\lambda \in (0, 1)$, put $y = \lambda x_1 + (1 - \lambda)x_2$. Since g_i is quasiconvex on $\mathbf{R}^n \times \mathbf{R}^k$, then for all $(x_1, a_1), (x_2, a_2) \in \mathbf{R}^n \times \mathbf{R}^k$, we have

$$g_i(\lambda x_1 + (1-\lambda)x_2, \lambda a_1 + (1-\lambda)a_2) \leq \max\{g_i(x_1, a_1), g_i(x_2, a_2)\}. \tag{8.24}$$

By (8.11) we get

$$[\tilde{X}_i]_\alpha = \{x \in \mathbf{R}^n \mid \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1\}.$$

Hence, it remains only to show that

$$\mu_{\tilde{R}_i}(g_i(\lambda x_1 + (1-\lambda)x_2; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1. \tag{8.25}$$

Apparently, from Proposition 6.52 and Corollary 6.53, it follows that $g_i(\lambda x_1 + (1-\lambda)x_2; [\tilde{a}_i]_\alpha)$ is a compact interval in $\mathbf{R}$. However, $[\tilde{b}_i]_\alpha$ is also a compact interval, therefore for $x \in \mathbf{R}^n$

$$\begin{aligned} \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) &= \sup\{\min\{\chi_{g_i(x;[\tilde{a}_i]_\alpha)}(u), \chi_{[\tilde{b}_i]_\alpha}(v)\} \mid u \le v\} \\ &= \begin{cases} 1 & \text{if } \underline{G}_i(x;\alpha) \le \overline{b}_i(\alpha), \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{8.26}$$

To prove (8.25), we have to show that for $y = \lambda x_1 + (1-\lambda)x_2$

$$\underline{G}_i(y;\alpha) \le \overline{b}_i(\alpha). \tag{8.27}$$

Observe that $[\tilde{a}_i]_\alpha$ is a convex and compact subset of $\mathbf{R}^k$. By continuity of g_i, there exists $a_j \in [\tilde{a}_i]_\alpha$, $j = 1, 2$, such that

$$\underline{G}_i(x_j;\alpha) = g_i(x_j; a_j). \tag{8.28}$$

Since $x_1, x_2 \in [\tilde{X}_i]_\alpha$, we get from (8.26) $\underline{G}_i(x_j;\alpha) \le \overline{b}_i(\alpha)$, $j = 1, 2$, or

$$\max\{\underline{G}_i(x_1;\alpha), \underline{G}_i(x_2;\alpha)\} \le \overline{b}_i(\alpha). \tag{8.29}$$

Then by (8.29) and (8.28) we immediately obtain

$$\max\{g_i(x_1; a_1), g_i(x_2; a_2)\} \le \overline{b}_i(\alpha). \tag{8.30}$$

Considering inequality (8.24), we get

$$g_i(y; \lambda a_1 + (1-\lambda)a_2) \le \max\{g_i(x_1; a_1), g_i(x_2; a_2)\}. \tag{8.31}$$

Apparently, as $\lambda a_1 + (1-\lambda)a_2 \in [\tilde{a}_i]_\alpha$, we get

$$\underline{G}_i(y;\alpha) \le g_i(y; \lambda a_1 + (1-\lambda)a_2). \tag{8.32}$$

Then inequalities (8.30) - (8.32) give the required result (8.27).

2. Let $i \in \mathcal{M}_1 \cup \mathcal{M}_3$, $\alpha \in (0, 1]$. Again, we show that $[\tilde{X}_i]_\alpha$ is convex. Let $x_1, x_2 \in [\tilde{X}_i]_\alpha$, $\lambda \in (0, 1)$, put $y = \lambda x_1 + (1-\lambda)x_2$. Since g_i is quasiconcave on $\mathbf{R}^n \times \mathbf{R}^k$, then for all $(x_1, a_1), (x_2, a_2) \in \mathbf{R}^n \times \mathbf{R}^k$, we have

$$g_i(\lambda x_1 + (1-\lambda)x_2, \lambda a_1 + (1-\lambda)a_2) \ge \min\{g_i(x_1, a_1), g_i(x_2, a_2)\}.$$

By (8.11) we get

$$[\tilde{X}_i]_\alpha = \{x \in \mathbf{R}^n \mid \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1\}.$$

Hence, it remains to show that

$$\mu_{\tilde{R}_i}(g_i(\lambda x_1 + (1-\lambda)x_2; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1. \qquad (8.33)$$

From Proposition 6.52 and Corollary 6.53, $g_i(\lambda x_1 + (1-\lambda)x_2; [\tilde{a}_i]_\alpha)$ is a compact interval in $\mathbf{R}$. However, $[\tilde{b}_i]_\alpha$ is also a compact interval, therefore for $x \in \mathbf{R}^n$

$$\begin{aligned} \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) &= \sup\{\min\{\chi_{g_i(x;[\tilde{a}_i]_\alpha)}(u), \chi_{[\tilde{b}_i]_\alpha}(v)\} \mid u \geq v\} \\ &= \begin{cases} 1 & \text{if } \overline{G}_i(x;\alpha) \geq \underline{b}_i(\alpha), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

To prove (8.33), we have to show that for $y = \lambda x_1 + (1-\lambda)x_2$

$$\overline{G}_i(y;\alpha) \geq \underline{b}_i(\alpha). \qquad (8.34)$$

The proof of (8.34) can be conducted in an analogous way to part 1. Observe first that $[\tilde{a}_i]_\alpha$ is a convex and compact subset of $\mathbf{R}^k$. By continuity of g_i, there exists $a'_j \in [\tilde{a}_i]_\alpha$, $j = 1, 2$, such that

$$\overline{G}_i(x_j;\alpha) = g_i(x_j; a'_j).$$

Since $x_1, x_2 \in [\tilde{X}_i]_\alpha$, we get from (8.26)

$$\overline{G}_i(x_j;\alpha) \geq \underline{b}_i(\alpha), \quad j = 1, 2,$$

or

$$\min\{\overline{G}_i(x_1;\alpha), \overline{G}_i(x_2;\alpha)\} \geq \underline{b}_i(\alpha).$$

Then by (8.29) and (8.29) we immediately obtain

$$\min\{g_i(x_1;a_1), g_i(x_2;a_2)\} \geq \underline{b}_i(\alpha). \qquad (8.35)$$

Considering inequality (8.24), we get

$$g_i(y; \lambda a_1 + (1-\lambda)a_2) \geq \min\{g_i(x_1;a_1), g_i(x_2;a_2)\}. \qquad (8.36)$$

Since $\lambda a_1 + (1-\lambda)a_2 \in [\tilde{a}_i]_\alpha$, we have

$$\overline{G}_i(y;\alpha) \geq g_i(y; \lambda a_1 + (1-\lambda)a_2). \qquad (8.37)$$

Then inequalities (8.35) - (8.37) give the required result (8.34). ∎

The main results of this section are schematically summarized in Table 8.1.

Table 8.1.

Constraint functions: g_i	Parameters: $\tilde{a}_i, \tilde{b}_i$	Relations: $R_i/\tilde{R}_i$	t-norm/ agr. op.	Results:	Theorem:
—	crisp	$=, \leq, \geq$ / fuzzy extension	T/T	$\tilde{X} = [\tilde{X}_i]_\alpha = \bar{X}$	T8.2
—	$\tilde{a}_i' \subset \tilde{a}_i''$, $\tilde{b}_i' \subset \tilde{b}_i''$	valued relat./ T-f. extens.	T/A	$\tilde{X}' \subset \tilde{X}''$	T8.3
continuous	compact	valued relat./ T-f. extens.	min / min	$[\tilde{X}]_\alpha = \bigcap_{i=1}^m [\tilde{X}_i]_\alpha$ $[\tilde{X}_i]_\alpha = \{x \in \mathbf{R}^n \mid \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) \geq \alpha\}$	T8.5
continuous	compact	$=, \leq, \geq$/ T-f. extens.	min / min	$[\tilde{X}]_\alpha = \bigcap_{i=1}^m [\tilde{X}_i]_\alpha$ $[\tilde{X}_i]_\alpha = \{x \in \mathbf{R}^n \mid \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1\}$	T8.6
continuous	compact UQCN	$=, \leq, \geq$ / T-f. extens.	min / min	$\underline{G}_i(x;\alpha) \leq \bar{b}_i(\alpha)$, $i \in \mathcal{M}_1 \cup \mathcal{M}_2$, $\overline{G}_i(x;\alpha) \geq \underline{b}_i(\alpha)$, $i \in \mathcal{M}_1 \cup \mathcal{M}_3$	T8.7
continuous QCV/ QCA	compact UQCN	$=, \leq, \geq$/ T-f. extens.	min / min	$[\tilde{X}_i]_\alpha$- convex	T8.8

7. Optimal Solutions of the FMP Problem

For convenience of the reader we first recall FMP problem (8.3). Let us consider an optimization problem associated with the MP problem (8.2), particularly, let $f, g_i, i \in \mathcal{M}$, be functions, $f : \mathbf{R}^n \times \mathbf{C} \to \mathbf{R}$, $g_i : \mathbf{R}^n \times \mathbf{P}_i \to \mathbf{R}$, where $\mathbf{C}$ and $\mathbf{P}_i$ are sets of parameters. Let $\mu_{\tilde{c}} : \mathbf{C} \to [0,1]$, $\mu_{\tilde{a}_i} : \mathbf{P}_i \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$ be membership functions of fuzzy parameters $\tilde{c}$, $\tilde{a}_i$ and $\tilde{b}_i$, respectively. Moreover, let $\tilde{R}_i$, $i \in \{0\} \cup \mathcal{M}$, be fuzzy relations with the corresponding membership functions $\mu_{\tilde{R}_i} : \mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}) \to [0,1]$.

Moreover, we assume that an additional fuzzy quantity $\tilde{b}_0$, called the *fuzzy goal*, is given with which the fuzzy values $\tilde{f}(x, \tilde{c})$ of the objective function are compared by means of a fuzzy relation $\tilde{R}_0$. Recall that both the fuzzy goal

and fuzzy relation $\tilde{R}_0$ are given exogenously, and cannot be derived from the data of problem (8.2). The fuzzy objective is then treated as another constraint $\tilde{f}(x;\tilde{c})\tilde{R}_0\tilde{b}_0$, and the maximization of $\tilde{f}(x;\tilde{c})$ means finding of a global maximizer of the membership function $\mu_{\tilde{R}_0}(\tilde{f}(x;\tilde{c}),\tilde{b}_0)$ over $\mathbf{R}^n$.

In this way we obtain the following formulation of the FMP problem associated with MP problem (8.2): Given a fuzzy goal $\tilde{b}_0$ and a fuzzy relation $\tilde{R}_0$,

$$\begin{aligned} &\text{maximize} && \tilde{f}(x;\tilde{c}) \\ &\text{subject to} && \tilde{g}_i(x;\tilde{a}_i)\tilde{R}_i\tilde{b}_i, \quad i \in \mathcal{M}. \end{aligned} \tag{8.38}$$

Its exact meaning is given by the following modification of Definition 8.1, where we assume that the set $\mathbf{C}$ and sets $\mathbf{P}_i$, $i \in \mathcal{M}$, together with functions $f : \mathbf{R}^n \times \mathbf{C} \to [0,1]$ and $g_i : \mathbf{R}^n \times \mathbf{P}_i \to [0,1]$, $i \in \mathcal{M}$, are given by problem (8.2).

DEFINITION 8.9 *For each $i \in \{0\}\cup\mathcal{M}$, let $\tilde{b}_i$ be a fuzzy quantity, $\tilde{R}_i$ be fuzzy relations with membership function $\mu_{\tilde{R}_i} : \mathcal{F}(\mathbf{R})\times\mathcal{F}(\mathbf{R}) \to [0,1]$. For $i \in \mathcal{M}$, let $\tilde{a}_i$ be a fuzzy subset of $\mathbf{P}_i$, and let $\tilde{c}$ be a fuzzy subset of $\mathbf{C}$. Moreover, let A and A_G be aggregation operators. The fuzzy subset $\tilde{X}^*$ of $\mathbf{R}$ whose membership function is given by*

$$\mu^*_{\tilde{X}}(x) = A_G(\mu_{\tilde{R}_0}(\tilde{f}(x;\tilde{c}),\tilde{b}_0),\mu_{\tilde{X}}(x)), \tag{8.39}$$

where $\mu_{\tilde{X}}$ is the membership function of the feasible solution given by (8.6), is called the optimal solution *of the FMP problem (8.38). For $\alpha \in (0,1]$, the elements of α-cut $[\tilde{X}^*]_\alpha$ are called the* α-optimal solutions *of the FMP problem (8.38), and the points $x \in \mathbf{R}^n$ with the property*

$$\mu^*_{\tilde{X}}(x^*) = \mathrm{Hgt}(\tilde{X}^*) \tag{8.40}$$

are called the max-optimal solutions.

Notice that the optimal solution $\tilde{X}^*$ of a FMP problem is a fuzzy subset of $\mathbf{R}^n$. Moreover, $\tilde{X}^* \subset \tilde{X}$, where $\tilde{X}$ is the feasible solution. On the other hand, the α-optimal solution is a point in $\mathbf{R}^n$, as well as the max-optimal solution, which is, in fact, the α-optimal solution with $\alpha = \mathrm{Hgt}(\tilde{X}^*)$. Notice that according to Chapter 7 a max-optimal solution is the max-A_G decision on $\mathbf{R}^n$.

In Definition 8.9 two aggregation operators A and A_G are used. The former is used for aggregating the individual constraints into the feasible solution by Definition 8.1, the latter is used for aggregating the fuzzy set of feasible solution given by the membership function

$$\mu_{\tilde{X}}(x) = A(\mu_{\tilde{R}_1}(\tilde{g}_1(x;\tilde{a}_1),\tilde{b}_1),\ldots,\mu_{\tilde{R}_m}(\tilde{g}_m(x;\tilde{a}_m),\tilde{b}_m))$$

with the fuzzy set "of the objective" $\tilde{X}_0$ defined by the membership function

$$\mu_{\tilde{X}_0}(x) = \mu_{\tilde{R}_0}(\tilde{f}(x;\tilde{c}), \tilde{b}_0). \tag{8.41}$$

As a result, we obtain the membership function of optimal solution $\tilde{X}^*$ as

$$\mu^*_{\tilde{X}}(x) = A_G(\mu_{\tilde{X}_0}(x), \mu_{\tilde{X}}(x)) \tag{8.42}$$

for all $x \in \mathbf{R}^n$. In particular, if $A = A_G$, then by commutativity and associativity we obtain (8.39) in a simple form

$$\mu^*_{\tilde{X}}(x) = A(\mu_{\tilde{R}_0}(\tilde{f}(x;\tilde{c}), \tilde{b}_0), \mu_{\tilde{R}_1}(\tilde{g}_1(x;\tilde{a}_1), \tilde{b}_1), \ldots, \mu_{\tilde{R}_m}(\tilde{g}_m(x;\tilde{a}_m), \tilde{b}_m)).$$

Since the problem (8.38) is a maximization problem, i.e., "the higher value is better", the membership function $\mu_{\tilde{b}_0}$ of $\tilde{b}_0$ should be increasing, or nondecreasing. By the same reason, the fuzzy relation $\tilde{R}_0$ for comparing $\tilde{f}(x;\tilde{c})$ and $\tilde{b}_0$ should be of the "greater or equal" type. In this section, we consider $\tilde{R}_0$ as a T-fuzzy extension of the usual binary operation $\geq$, where T is a t-norm.

Notice that our approach can be extended to a multi-objective MP problem with more than one objective functions and more fuzzy goals and corresponding fuzzy relations.

Formally, in Definitions 8.1 and 8.9, the concepts of feasible solution and optimal solution, α-feasible solution and α-optimal solution, respectively, are similar to each other. Therefore, we can take advantage of the results derived already in the preceding section. First we show that in the case of crisp parameters c, a_i and b_i, the max-optimal solution given by (8.40) coincides with the optimal solution of the crisp problem.

THEOREM 8.10 *Let* $f : \mathbf{R}^n \times \mathbf{C} \to \mathbf{R}$, $g_i : \mathbf{R}^n \times \mathbf{P}_i \to \mathbf{R}$, *let* $c \in \mathbf{C}$, $a_i \in \mathbf{P}_i$ *and* $b_i \in \mathbf{R}$ *be crisp parameters,* $i \in \mathcal{M}$. *Let* $\tilde{b}_0 \in \mathcal{F}(\mathbf{R})$ *be a fuzzy goal with the strictly increasing membership function* $\mu_{\tilde{b}_0} : \mathbf{R} \to [0,1]$, *let* $\tilde{R}_i$, $i \in \{0\} \cup \mathcal{M}$, *be fuzzy relations such that, for* $i \in \mathcal{M}_1$, $\tilde{R}_i$ *is a fuzzy extension of the equality relation* "$=$", *for* $i \in \mathcal{M}_2$, $\tilde{R}_i$ *is a fuzzy extension of the inequality relation* "$\leq$", *and for* $i \in \{0\} \cup \mathcal{M}_3$, $\tilde{R}_i$ *is a fuzzy extension of the inequality relation* "$\geq$". *Let* T, A *and* A_G *be t-norms.*

Then the set of all max-optimal solutions coincides with the set of all optimal solutions X^* *of the corresponding instance of MP problem (8.2).*

PROOF. By Theorem 8.2, the feasible solution of (8.38) is crisp. Therefore

$$\mu_{\tilde{X}}(x) = \chi_X(x) \tag{8.43}$$

for all $x \in \mathbf{R}^n$, where X is the set of all feasible solutions of the corresponding crisp MP problem.

Moreover, by (8.41) for crisp $c \in \mathbf{C}$ we obtain

$$\begin{aligned} \mu_{\tilde{X}_0}(x) &= \mu_{\tilde{R}_0}(\tilde{f}(x;\tilde{c}), \tilde{b}_0) \\ &= \sup\{T(\chi_{f(x,c)}(u), \mu_{\tilde{b}_0}(v)) \mid u \geq v\} \\ &= \mu_{\tilde{b}_0}(f(x,c)). \end{aligned} \tag{8.44}$$

Substituting (8.43) and (8.44) into (8.42), we obtain

$$\mu^*_{\tilde{X}}(x) = A_G(\mu_{\tilde{b}_0}(f(x,c)), \chi_X(x)) = \begin{cases} \mu_{\tilde{b}_0}(f(x,c)) & \text{if } x \in X, \\ 0 & \text{otherwise.} \end{cases} \tag{8.45}$$

Since $\mu_{\tilde{b}_0}$ is strictly increasing function, it follows from (8.45) that

$$\mu^*_{\tilde{X}}(x^*) = \mathrm{Hgt}(\tilde{X}^G),$$

if and only if

$$\mu^*_{\tilde{X}}(x^*) = \sup\{\mu_{\tilde{b}_0}(f(x,c)) \mid x \in X\},$$

which is the desired result. ∎

Recall, that for fuzzy subsets $\tilde{a}', \tilde{a}'' \in \mathcal{F}(\mathbf{R}^n)$, we have $\tilde{a}' \subset \tilde{a}''$ if and only if $\mu_{\tilde{a}'}(x) \leq \mu_{\tilde{a}''}(x)$ for all $x \in \mathbf{R}^n$.

THEOREM 8.11 *Let f, g_i, $i \in \mathcal{M}$, be real-valued functions, $f : \mathbf{R}^n \times \mathbf{C} \to \mathbf{R}$, $g_i : \mathbf{R}^n \times \mathbf{P}_i \to \mathbf{R}$. Let $\tilde{c}'$, $\tilde{a}'_i$, $\tilde{b}'_i$, and $\tilde{c}''$, $\tilde{a}''_i$, $\tilde{b}''_i$ be two collections of fuzzy parameters of the FMP problem. Let T, A and A_G be t-norms. Let $\tilde{b}_0 \in \mathcal{F}(\mathbf{R})$ be a fuzzy goal, let $\tilde{R}_i = \Psi^T(R_i)$, $i \in \{0\} \cup \mathcal{M}$, i.e., $\tilde{R}_i$ be T-fuzzy extension of valued relation R_i on $\mathbf{R}$.*

If $\tilde{X}^{*\prime}$ is the optimal solution of FMP problem (8.38) with the parameters $\tilde{c}'$, $\tilde{a}'_i$ and $\tilde{b}'_i$, and $\tilde{X}^{*\prime\prime}$ is the optimal solution of the FMP problem with the parameters $\tilde{c}''$, $\tilde{a}''_i$ and $\tilde{b}''_i$ such that for all $i \in \mathcal{M}$

$$\tilde{c}' \subset \tilde{c}'', \ \tilde{a}'_i \subset \tilde{a}''_i \text{ and } \tilde{b}'_i \subset \tilde{b}''_i,$$

then

$$\tilde{X}^{*\prime} \subset \tilde{X}^{*\prime\prime}.$$

PROOF. By Theorem 8.3, for the corresponding feasible solutions, it holds $\tilde{X}' \subset \tilde{X}''$. It remains to show that $\tilde{X}'_0 \subset \tilde{X}''_0$, where

$$\mu_{\tilde{X}'_0}(x) = \mu_{\tilde{R}_0}(\tilde{f}(x;\tilde{c}'), \tilde{b}_0), \ \mu_{\tilde{X}''_0}(x) = \mu_{\tilde{R}_0}(\tilde{f}(x;\tilde{c}''), \tilde{b}_0).$$

First, we show that $\tilde{f}(x;\tilde{c}') \subset \tilde{f}(x;\tilde{c}'')$.

Indeed, since $\mu_{\tilde{c}'}(c) \leq \mu_{\tilde{c}''}(c)$ for all $c \in \mathbf{C}$, by (8.5), we obtain for all $u \in \mathbf{R}$

$$\begin{aligned}\mu_{\tilde{f}(x;\tilde{c}')}(u) &= \max\{0, \sup\{\mu_{\tilde{c}'}(c) \mid c \in \mathbf{C},\, f(x;c) = u\}\} \\ &\leq \max\{0, \sup\{\mu_{\tilde{c}''}(c) \mid c \in \mathbf{C},\, f(x;c) = u\}\} \\ &= \mu_{\tilde{f}(x;\tilde{c}'')}(u).\end{aligned}$$

Now, using monotonicity of T-fuzzy extension $\tilde{R}_0$, it follows that

$$\mu_{\tilde{R}_i}(\tilde{f}(x;\tilde{c}'), \tilde{b}_0) \leq \mu_{\tilde{R}_i}(\tilde{f}(x;\tilde{c}''), \tilde{b}_0).$$

Applying again monotonicity of A_G in (8.42), we finally obtain $\tilde{X}^{*\prime} \subset \tilde{X}^{*\prime\prime}$. ■

COROLLARY 8.12 *Let $\tilde{c}$, $\tilde{a}_i$, $\tilde{b}_i$ be a collection of fuzzy parameters, and let $c \in \mathbf{C}$, $a_i \in \mathbf{P}_i$ and $b_i \in \mathbf{R}$ be a collection of crisp parameters such that for all $i \in \mathcal{M}$*

$$\mu_{\tilde{c}}(c) = \mu_{\tilde{a}_i}(a_i) = \mu_{\tilde{b}_i}(b_i) = 1. \tag{8.46}$$

If X^* is a nonempty set of all optimal solutions of MP problem (8.2) with the parameters c, a_i and b_i, and $\tilde{X}^*$ is the optimal solution of FMP problem (8.38) with fuzzy parameters $\tilde{c}$, $\tilde{a}_i$ and $\tilde{b}_i$, then for all $x \in X^*$

$$\mu^*_{\tilde{X}}(x) = 1. \tag{8.47}$$

PROOF. Observe that $c \subset \tilde{c}$, $a_i \subset \tilde{a}_i$, $b_i \subset \tilde{b}_i$ for all $i \in \mathcal{M}$. By Theorem 8.11 we obtain $X^* \subset \tilde{X}^*$, which is equivalent to (8.47). ■

Notice that the optimal solution $\tilde{X}^*$ of FMP problem (8.3) always exists, even if the MP problem with crisp parameters has no crisp optimal solution. Corollary 8.12 states that if the MP problem with crisp parameters has a crisp optimal solution, then the membership grade of the optimal solution (of the associated FMP problem with fuzzy parameters) is equal to one. This fact guarantees that the class of MP problems can be naturally be embedded into the class of FMP problems.

In the remainder of this section, we assume that all parameter sets $\mathbf{C}$ and $\mathbf{P}_i$, $i \in \mathcal{M}$, are equal to the k-dimensional vector space $\mathbf{R}^k$. Also we assume that t-form T together with aggregation operators A and A_G are all equal to the minimum t-norm T_M.

In Theorem 8.13 - 8.16, we present some formulae and results based on α-cuts of the parameters that are analogous to those given by Theorem 8.5 - 8.8, however, now for α-optimal solutions of the FMP problem. We omit the proofs because they are analogous to those of Theorem 8.5 - 8.8.

THEOREM 8.13 *Let f and g_i, $i \in \mathcal{M}$, be continuous real-valued functions defined on $\mathbf{R}^n \times \mathbf{R}^k$. Let $\tilde{c}$, $\tilde{a}_i$ and $\tilde{b}_i$ be compact fuzzy parameters, $i \in \mathcal{M}$, let $\tilde{b}_0 \in \mathcal{F}(\mathbf{R})$ be a fuzzy goal. Let $\tilde{R}_i = \Psi^T(R_i)$ be T-fuzzy extensions of closed valued relations R_i on $\mathbf{R}$, $i \in \{0\} \cup \mathcal{M}$.*

Then for all $\alpha \in (0, 1]$

$$[\tilde{X}^*]_\alpha = \bigcap_{i=0}^{m} [\tilde{X}_i]_\alpha,$$

and, moreover, for all $i \in \mathcal{M}$ we have

$$\begin{aligned} [\tilde{X}_0]_\alpha &= \{x \in \mathbf{R}^n \mid \mu_{\tilde{R}_0}(f(x; [\tilde{c}]_\alpha), [\tilde{b}_0]_\alpha) \geq \alpha\}, \\ [\tilde{X}_i]_\alpha &= \{x \in \mathbf{R}^n \mid \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) \geq \alpha\}. \end{aligned}$$

If the valued relations R_i are usual equality and inequality relations, then a stronger statement can be proved.

THEOREM 8.14 *Let f and g_i, $i \in \mathcal{M}$, be continuous real-valued functions defined on $\mathbf{R}^n \times \mathbf{R}^k$. Let $\tilde{c}$, $\tilde{a}_i$ and $\tilde{b}_i$ be compact fuzzy parameters. Let $\tilde{R}_i$ be the same as in Theorem 8.10, $i \in \{0\} \cup \mathcal{M}$.*

Then for all $\alpha \in (0, 1]$

$$[\tilde{X}^*]_\alpha = \bigcap_{i=0}^{m} [\tilde{X}_i]_\alpha,$$

and, moreover, for all $i \in \mathcal{M}$ we have

$$\begin{aligned} [\tilde{X}_0]_\alpha &= \{x \in \mathbf{R}^n \mid \mu_{\tilde{R}_0}(f(x; [\tilde{c}]_\alpha), [\tilde{b}_0]_\alpha) = 1\}, \\ [\tilde{X}_i]_\alpha &= \{x \in \mathbf{R}^n \mid \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1\}. \end{aligned}$$

Next, we present an analogue to Theorem 8.7. For this purpose we extend the notation from the previous section as follows. Given $\alpha \in (0, 1]$, $i \in \mathcal{M}$, let

$$\begin{aligned} \overline{F}(x; \alpha) &= \sup\{f(x; c) \mid c \in [\tilde{c}]_\alpha\}, \\ \underline{F}(x; \alpha) &= \inf\{f(x; c) \mid c \in [\tilde{c}]_\alpha\}, \\ \underline{b}_0(\alpha) &= \inf\{b \in \mathbf{R} \mid b \in [\tilde{b}_0]_\alpha\}, \\ \overline{b}_0(\alpha) &= \sup\{b \in \mathbf{R} \mid b \in [\tilde{b}_0]_\alpha\}. \end{aligned}$$

THEOREM 8.15 *Let all assumptions of Theorem 8.14 be satisfied. Moreover, let the membership functions of fuzzy parameters $\tilde{c}, \tilde{a}_i$ and $\tilde{b}_i$ be upper-quasiconnected for all $i \in \mathcal{M}$. Let $\tilde{b}_0 \in \mathcal{F}(\mathbf{R})$ be a fuzzy goal with the membership function $\mu_{\tilde{b}_0}$ satisfying the following conditions*

$$\begin{aligned} &\mu_{\tilde{b}_0} \text{ is upper semicontinuous,} \\ &\mu_{\tilde{b}_0} \text{ is strictly increasing,} \\ &\lim_{t\to-\infty} \mu_{\tilde{b}_0}(t) = 0. \end{aligned} \tag{8.48}$$

Then, for all $\alpha \in (0,1]$, we have $x \in [\tilde{X}^*]_\alpha$ if and only if

$$\begin{aligned} \overline{F}(x;\alpha) &\geq \underline{b}_0(\alpha), \\ \underline{G}_i(x;\alpha) &\leq \overline{b}_i(\alpha), \quad i \in \mathcal{M}_1 \cup \mathcal{M}_2, \\ \overline{G}_i(x;\alpha) &\geq \underline{b}_i(\alpha), \quad i \in \mathcal{M}_1 \cup \mathcal{M}_3, \end{aligned}$$

Again, the proof is omitted, since it is analogous to the proof of Theorem 8.7 with the only modification that instead of compactness of $\tilde{b}_0$, we have assumptions (8.48).

THEOREM 8.16 *Let all assumptions of Theorem 8.14 be satisfied. Moreover, let g_i be quasiconvex on $\mathbf{R}^n \times \mathbf{R}^k$ for $i \in \mathcal{M}_1 \cup \mathcal{M}_2$, f and g_i be quasiconcave on $\mathbf{R}^n \times \mathbf{R}^k$ for $i \in \mathcal{M}_1 \cup \mathcal{M}_3$.*

Then for all $i \in \{0\} \cup \mathcal{M}$, $\tilde{X}_i$ are convex and the optimal solution $\tilde{X}^*$ of FMP problem (8.38) is convex too.

The last result of this section says that if the individual membership functions of the fuzzy objective and fuzzy constraints can be expressed explicitly, then the max-optimal solution can be found as the optimal solution of some crisp MP problem.

THEOREM 8.17 *Let*

$$\mu_{\tilde{X}_0}(x) = \mu_{\tilde{R}_0}(\tilde{f}(x;\tilde{c}), \tilde{b}_0)$$

and

$$\mu_{\tilde{X}_i}(x) = \mu_{\tilde{R}_i}(\tilde{g}_i(x;\tilde{a}_i), \tilde{b}_i),$$

$x \in \mathbf{R}^n$, $i \in \mathcal{M}$, be the membership functions of the fuzzy objective and fuzzy constraints of the FMP problem (8.38), respectively. Let (8.48) hold for $\tilde{b}_0$.

Then the vector $(t^*, x^*) \in \mathbf{R}^{n+1}$ is an optimal solution of the problem

$$\begin{aligned} &\text{maximize} \quad t \\ &\text{subject to} \quad \mu_{\tilde{X}_i}(x) \geq t, \quad i \in \{0\} \cup \mathcal{M}, \end{aligned} \tag{8.49}$$

if and only if x^* is the max-optimal solution of the problem (8.38).

PROOF. Let $(t^*, x^*) \in \mathbf{R}^{n+1}$ be an optimal solution of the MP problem (8.49). By (8.40) and (8.42) we obtain

$$\mu^*_{\tilde{X}}(x^*) = \sup\{\min\{\mu_{\tilde{X}_0}(x), \mu_{\tilde{X}_i}(x)\} \mid x \in \mathbf{R}^n\} = \mathrm{Hgt}(\tilde{X}^*).$$

Hence, x^* is an max-optimal solution. The proof of the converse statement is straightforward. ∎

Chapter 9

FUZZY LINEAR PROGRAMMING

1. Introduction

Most frequent mathematical programming problems are linear programming problems. In this chapter we are concerned with fuzzy linear programming problem related to linear programming problems in the following form.

Let $\mathcal{M} = \{1, 2, \ldots, m\}$ and $\mathcal{N} = \{1, 2, \ldots, n\}$ where m and n are positive integers, and let all parameter sets $\mathbf{C}$ and $\mathbf{P}_i$, $i \in \mathcal{M}$, be equal to $\mathbf{R}^n$. Then for each $c = (c_1, c_2, \ldots, c_n)$ and $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$, $i \in \mathcal{M}$, the functions $f(\cdot; c)$ and $g(\cdot; a_i)$ defined on $\mathbf{R}^n$ by

$$\begin{aligned} f(x; c_1, \ldots, c_n) &= c_1x_1 + \cdots + c_nx_n, && (9.1)\\ g_i(x; a_{i1}, \ldots, a_{in}) &= a_{i1}x_1 + \cdots + a_{in}x_n, \quad i \in \mathcal{M}, && (9.2) \end{aligned}$$

are linear on $\mathbf{R}^n$. For each $c \in \mathbf{R}^n$ and $a_i \in \mathbf{R}^n$, $i \in \mathcal{M}$, we consider the linear programming problem (LP)

$$\begin{array}{ll} \text{maximize} & c_1x_1 + \cdots + c_nx_n \\ \text{subject to} & a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i, \quad i \in \mathcal{M}, \\ & x_j \geq 0, \quad j \in \mathcal{N}. \end{array} \tag{9.3}$$

The set of all feasible solutions of problem (9.3) is denoted by X, that is,

$$X = \{x \in \mathbf{R}^n \mid a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i,\ i \in \mathcal{M},\ x_j \geq 0,\ j \in \mathcal{N}\}. \tag{9.4}$$

2. Formulation of FLP problem

Before formulating a fuzzy linear problem as an optimization problem associated with the LP problem (9.3), we make a few assumptions and remarks.

Let f, g_i be linear functions defined by (9.1), (9.2), respectively. From now on, the parameters c_j, a_{ij} and b_i will be considered as normal *fuzzy quantities*, that is, normal fuzzy subsets of the Euclidean space $\mathbf{R}$. The fuzzy quantities will be denoted by symbols with the tilde. Let $\mu_{\tilde{c}_j} : \mathbf{R} \to [0,1]$, $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, be membership functions of the fuzzy parameters $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$, respectively.

Let $\tilde{R}_i$, $i \in \mathcal{M}$, be fuzzy relations on $\mathcal{F}(\mathbf{R})$. They will be used for comparing the left and right sides of the constraints.

The maximization of an objective function needs, however, a special treatment, similar to that of FMP problem. As it was stated in Chapter 8, the set of fuzzy values of the objective function is not linearly ordered and to maximize the objective function we have to define a suitable ordering on $\mathcal{F}(\mathbf{R})$ which allows for "maximization" of the objective. Again it shall be done by an exogenously given fuzzy goal $\tilde{d} \in \mathcal{F}(\mathbf{R})$ and another fuzzy relation $\tilde{R}_0$ on $\mathbf{R}$. There exist some other approaches, see [26], [32], [84].

The *fuzzy linear programming problem (FLP problem)* associated with LP problem (9.3) is written in the form

$$\begin{array}{lll} \text{maximize} & \tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n & \\ \text{subject to} & (\tilde{a}_{i1} x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n) \tilde{R}_i \tilde{b}_i, \quad i \in \mathcal{M}, & \\ & x_j \geq 0, \quad j \in \mathcal{N}. & \end{array} \tag{9.5}$$

Let us clarify the elements of (9.5).

The objective function values and the left hand sides values of the constraints of (9.5) are obtained by the extension principle (6.15) as follows. The membership function of $\tilde{g}_i(x; \tilde{a}_{i1}, \dots, \tilde{a}_{i1})$ is defined for each $t \in \mathbf{R}$ by

$$\mu_{\tilde{g}_i}(t) = \begin{cases} \sup \left\{ T(\mu_{\tilde{a}_{i1}}(a_1), \dots, \mu_{\tilde{a}_{in}}(a_n)) \;\middle|\; \begin{array}{l} a_1, \dots, a_n \in \mathbf{R}, \\ a_1 x_1 + \cdots + a_n x_n = t \end{array} \right\} & \\ & \text{if } g_i^{-1}(x;t) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$g_i^{-1}(x;t) = \{(a_1, \dots, a_n) \in \mathbf{R}^n \mid a_1 x_1 + \cdots + a_n x_n = t\}.$$

Here, the fuzzy set $\tilde{g}_i(x; \tilde{a}_{i1}, \dots, \tilde{a}_{i1})$ is denoted as $\tilde{a}_{i1} x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n$, i.e.,

$$\tilde{g}_i(x; \tilde{a}_{i1}, \dots, \tilde{a}_{i1}) = \tilde{a}_{i1} x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n$$

for every $i \in \mathcal{M}$ and for each $x \in \mathbf{R}^n$.

Similarly, for given $\tilde{c}_1, \dots, \tilde{c}_n \in \mathcal{F}(\mathbf{R})$, $\tilde{f}(x; \tilde{c}_1, \dots, \tilde{c}_n)$ is the fuzzy extension of $f(x; c_1, \dots, c_n)$ with the membership function defined for each $t \in \mathbf{R}$

by

$$\mu_{\tilde{f}}(t) = \begin{cases} \sup\left\{T(\mu_{\tilde{c}_1}(c_1), \ldots, \mu_{\tilde{c}_n}(c_n)) \;\middle|\; \begin{array}{l} c_1, \ldots, c_n \in \mathbf{R}, \\ c_1x_1 + \cdots + c_nx_n = t \end{array}\right\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } f^{-1}(x;t) \neq \emptyset, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; \text{otherwise,} \end{cases} \tag{9.6}$$

where $f^{-1}(x;t) = \{(c_1, \ldots, c_n) \in \mathbf{R}^n \mid f(x; c_1, \ldots, c_n) = t\}$.

The fuzzy set $\tilde{f}(x; \tilde{c}_1, \ldots, \tilde{c}_n)$ will be denoted as $\tilde{c}_1x_1\tilde{+}\cdots\tilde{+}\tilde{c}_nx_n$, i.e.,

$$\tilde{f}(x; \tilde{c}_1, \ldots, \tilde{c}_n) = \tilde{c}_1x_1\tilde{+}\cdots\tilde{+}\tilde{c}_nx_n.$$

In (9.5) the value $\tilde{a}_{i1}x_1\tilde{+}\cdots\tilde{+}\tilde{a}_{in}x_n \in \mathcal{F}(\mathbf{R})$ is "compared" with the fuzzy quantity $\tilde{b}_i \in \mathcal{F}(\mathbf{R})$ by fuzzy relation $\tilde{R}_i$, $i \in \mathcal{M}$.

As mentioned above, we have to define a suitable ordering on $\mathcal{F}(\mathbf{R})$ for the comparison of fuzzy quantities $\tilde{f}(x; \tilde{c}_1, \ldots, \tilde{c}_n) \in \mathcal{F}(\mathbf{R})$, $x \in \mathbf{R}^n$. Let $\tilde{d} \in \mathcal{F}(\mathbf{R})$ and $\tilde{R}_0$ be exogenously given fuzzy goal and fuzzy relation on $\mathcal{F}(\mathbf{R})$, respectively.

The fuzzy relations $\tilde{R}_i$ for comparing the constraints of (9.5) are usually extensions of a valued relation on $\mathbf{R}$, particularly, the usual inequality relations "$\leq$" and "$\geq$".

If $\tilde{R}_i = \Psi^T(R_i)$ is the T-fuzzy extension of relation $R_i \in \{\leq, \geq\}$, then by (6.25) we obtain the membership function of the ith constraint as

$$\mu_{\tilde{R}_i}(\tilde{a}_{i1}x_1\tilde{+}\cdots\tilde{+}\tilde{a}_{in}x_n, \tilde{b}_i) = \sup\{T(\mu_{\tilde{a}_{i1}x_1\tilde{+}\cdots\tilde{+}\tilde{a}_{in}x_n}(u), \mu_{\tilde{b}_i}(v)) \mid uR_iv\}.$$

Apparently, for the feasible solution and also for the optimal solution of a FLP problem, the concepts which have been defined in the preceding chapter for FMP problem (8.2), can be adopted here. Of course, for FLP problems they have some special features. Let us begin with the concept of feasible solution.

DEFINITION 9.1 *Let g_i, $i \in \mathcal{M}$, be functions defined by (9.2). Let $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, be membership functions of fuzzy quantities $\tilde{a}_{ij}$ and $\tilde{b}_i$, respectively. Let $\tilde{R}_i$, $i \in \mathcal{M}$, be fuzzy relations on $\mathcal{F}(\mathbf{R})$. Let G_A be an aggregation operator and T be a t-norm used for extending arithmetic operations.*

A fuzzy set $\tilde{X}$, the membership function $\mu_{\tilde{X}}$ of which is defined for all $x \in \mathbf{R}^n$ by

$$\mu_{\tilde{X}}(x) = \begin{cases} G_A(\mu_{\tilde{R}_1}(\tilde{a}_{11}x_1\cdots\tilde{+}\tilde{a}_{1n}x_n, \tilde{b}_1), & \\ \ldots, \mu_{\tilde{R}_m}(\tilde{a}_{m1}x_1\tilde{+}\cdots\tilde{+}\tilde{a}_{mn}x_n, \tilde{b}_m)) & \text{if } x_j \geq 0 \text{ for all } j \in \mathcal{N}, \\ 0 & \text{otherwise,} \end{cases} \tag{9.7}$$

is called the feasible solution *of the FLP problem (9.5).*

For $\alpha \in (0,1]$, *a vector* $x \in [\tilde{X}]_\alpha$ *is called the* α-feasible solution *of the FLP problem (9.5).*

A vector $\bar{x} \in \mathbf{R}^n$ *such that* $\mu_{\tilde{X}}(\bar{x}) = \mathrm{Hgt}(\tilde{X})$ *is called the* max-feasible solution.

Notice that the feasible solution $\tilde{X}$ of a FLP problem is a fuzzy set. On the other hand, the α-feasible solution is a vector belonging to the α-cut of the feasible solution $\tilde{X}$ and the same holds for the max-feasible solution, which is a special α-feasible solution with $\alpha = \mathrm{Hgt}(\tilde{X})$.

If a decision maker specifies the degree of membership $\alpha \in (0,1]$ (the degree of possibility, feasibility, satisfaction etc.), then a vector $x \in \mathbf{R}^n$ satisfying $\mu_{\tilde{X}}(x) \geq \alpha$ is the α-feasible solution of the corresponding FLP problem.

For $i \in \mathcal{M}$ we introduce the following notation: $\tilde{X}_i$ will denote the fuzzy subset of $\mathbf{R}^n$ with the membership function $\mu_{\tilde{X}_i}$ defined for all $x \in \mathbf{R}^n$ as

$$\mu_{\tilde{X}_i}(x) = \mu_{\tilde{R}_i}(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i). \tag{9.8}$$

Fuzzy set (9.8) is interpreted as ith fuzzy constraint. All fuzzy constraints are aggregated into the feasible solution (9.7) by the aggregation operator G_A. Particularly, $G_A = \min$, the t-norm T is used for extending arithmetic operations. Notice, that the fuzzy solution depends also on the fuzzy relations used in definitions of the constraints of the FLP problem.

3. Properties of Feasible Solution

For crisp parameters a_{ij} and b_i, clearly, the feasible solution is also crisp. Moreover, it is not difficult to show that if the fuzzy parameters of two FLP problems are ordered by fuzzy inclusion, that is, $\tilde{a}'_{ij} \subset \tilde{a}''_{ij}$ and $\tilde{b}'_i \subset \tilde{b}''_i$, then the same inclusion holds for the feasible solutions, i.e., $\tilde{X}' \subset \tilde{X}''$, on condition $\tilde{R}_i = \Psi^T(R_i)$ are T-fuzzy extensions of valued relations R_i; see also Proposition 9.5 below.

Now, we derive special formulae which will allow for computing an α-feasible solution $x \in [\tilde{X}]_\alpha$ of the FLP problem (9.5). For this purpose, the following notation will be useful. Given $\alpha \in (0,1], i \in \mathcal{M}, j \in \mathcal{N}$, let

$$\underline{a}_{ij}(\alpha) = \inf\{a \in \mathbf{R} \mid a \in [\tilde{a}_{ij}]_\alpha\}, \tag{9.9}$$
$$\overline{a}_{ij}(\alpha) = \sup\{a \in \mathbf{R} \mid a \in [\tilde{a}_{ij}]_\alpha\}, \tag{9.10}$$
$$\underline{b}_i(\alpha) = \inf\{b \in \mathbf{R} \mid b \in [\tilde{b}_i]_\alpha\}, \tag{9.11}$$
$$\overline{b}_i(\alpha) = \sup\{b \in \mathbf{R} \mid b \in [\tilde{b}_i]_\alpha\}. \tag{9.12}$$

THEOREM 9.2 *For all $i \in \mathcal{M}$ and $j \in \mathcal{N}$, let $\tilde{a}_{ij}$ and $\tilde{b}_i$ be compact, convex and normal fuzzy quantities, and let $x_j \geq 0$. Let $T = \min$, $S = \max$, and $\alpha \in (0,1)$.*

Then for $i \in \mathcal{M}$ we have

(i)

$$\mu_{\Psi^T(\leq)}(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i) \geq \alpha \text{ iff } \sum_{j=1}^{n} \underline{a}_{ij}(\alpha)x_j \leq \overline{b}_i(\alpha), \quad (9.13)$$

(ii)

$$\mu_{\Psi_S(\leq)}(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i) \geq \alpha \text{ iff } \sum_{j=1}^{n} \overline{a}_{ij}(1-\alpha)x_j \leq \underline{b}_i(1-\alpha). \quad (9.14)$$

(iii) *Moreover, if $\tilde{a}_{ij}$ and $\tilde{b}_i$ are strictly convex fuzzy quantities, then*

$$\mu_{\Psi^{T,S}(\leq)}(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i) \geq \alpha \text{ iff } \sum_{j=1}^{n} \overline{a}_{ij}(1-\alpha)x_j \leq \overline{b}_i(\alpha), \quad (9.15)$$

and $\mu_{\Psi^{T,S}(\leq)} = \mu_{\Psi_{T,S}(\leq)}$;

(iv)

$$\mu_{\Psi^{S,T}(\leq)}(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i) \geq \alpha \text{ iff } \sum_{j=1}^{n} \underline{a}_{ij}(\alpha)x_j \leq \underline{b}_i(1-\alpha), \quad (9.16)$$

and $\mu_{\Psi^{S,T}(\leq)} = \mu_{\Psi_{S,T}(\leq)}$.

PROOF. We present only a proof of part (i). The remaining parts follow analogically from Theorem 6.63.

Let $i \in \mathcal{M}$, $\mu_{\Psi^T(\leq)}(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i) \geq \alpha$. By Theorem 6.63 , this is equivalent to

$$\inf \left[\sum_{j=1}^{n} \tilde{a}_{ij}x_j \right]_{\alpha} \leq \sup[\tilde{b}_i]_{\alpha}.$$

From the well known Nguyen's result, see [60] or [88], it follows that

$$\left[\sum_{j=1}^{n} \tilde{a}_{ij}x_j \right]_{\alpha} = \sum_{j=1}^{n} [\tilde{a}_{ij}]_{\alpha} x_j.$$

Since $[\tilde{a}_{ij}]_\alpha$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, are compact and convex intervals in $\mathbf{R}$ and $x_j \geq 0$, $j \in \mathcal{N}$, the rest of the proof follows easily from definitions (9.9), (9.10) and Proposition 8.3. ■

Let $l, r, \in \mathbf{R}$ with $l \leq r$, let $\gamma, \delta \in [0, +\infty)$ and let L, R be non-increasing, non-constant, upper-semicontinuous functions mapping interval $(0, 1]$ into $[0, +\infty)$, i.e., $L, R : (0, 1] \to [0, +\infty)$. Moreover, assume that $L(1) = R(1) = 0$, define $L(0) = \lim_{x \to 0} L(x)$, $R(0) = \lim_{x \to 0} R(x)$.

Let A be an (L, R)-fuzzy interval given by the membership function defined by

$$\mu_A(x) = \begin{cases} L^{(-1)}\left(\frac{l-x}{\gamma}\right) & \text{if } x \in (l-\gamma, l), \gamma > 0, \\ 1 & \text{if } x \in [l, r], \\ R^{(-1)}\left(\frac{x-r}{\delta}\right) & \text{if } x \in (r, r+\delta), \delta > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $L^{(-1)}$, $R^{(-1)}$ are pseudo-inverse functions of L, R, respectively. As it was mentioned in Chapter 6, the class of (L, R)-fuzzy intervals extends the class of crisp closed intervals $[a, b] \subset \mathbf{R}$ including the case $a = b$, i.e., crisp numbers. Particularly, if the membership functions of $\tilde{a}_{ij}$ and $\tilde{b}_i$ are given analytically by

$$\mu_{\tilde{a}_{ij}}(x) = \begin{cases} L^{(-1)}\left(\frac{l_{ij}-x}{\gamma_{ij}}\right) & \text{if } x \in [l_{ij}-\gamma_{ij}, l_{ij}), \gamma_{ij} > 0, \\ 1 & \text{if } x \in [l_{ij}, r_{ij}], \\ R^{(-1)}\left(\frac{x-r_{ij}}{\delta_{ij}}\right) & \text{if } x \in (r_{ij}, r_{ij}+\delta_{ij}], \delta_{ij} > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mu_{\tilde{b}_i}(x) = \begin{cases} L^{(-1)}\left(\frac{l_i-x}{\gamma_i}\right) & \text{if } x \in [l_i-\gamma_i, l_i), \gamma_i > 0, \\ 1 & \text{if } x \in [l_i, r_i], \\ R^{(-1)}\left(\frac{x-r_i}{\delta_i}\right) & \text{if } x \in (r_i, r_i+\delta_i], \delta_i > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{9.17}$$

for each $x \in \mathbf{R}$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, then the values of (9.9) - (9.12) can be computed as

$$\begin{aligned} \underline{a}_{ij}(\alpha) &= l_{ij} - \gamma_{ij} L(\alpha), & \overline{a}_{ij}(\alpha) &= r_{ij} + \delta_{ij} R(\alpha), \\ \underline{b}_i(\alpha) &= l_i - \gamma_i L(\alpha), & \overline{b}_i(\alpha) &= r_i + \delta_i R(\alpha). \end{aligned}$$

Let $G_A = \min$. By Proposition 9.2, all α-cuts $[\tilde{X}]_\alpha$ of the feasible solution of (9.5) can be computed by solving the system of inequalities from (9.13) - (9.16). Moreover, $[\tilde{X}]_\alpha$ is the intersection of a finite number of halfspaces, hence a convex polyhedral set.

4. Properties of Optimal Solutions

As the FLP problem is a particular case of the FMP problem, all properties and results which have been derived in Chapter 8 are applicable to FLP problems.

We assume the existence of an exogenously given additional goal $\tilde{d} \in \mathcal{F}(\mathbf{R})$. The fuzzy value $\tilde{d}$ is compared to fuzzy values $\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n$ of the objective function by a given fuzzy relation $\tilde{R}_0$. In this way the fuzzy objective is treated as another constraint

$$\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n \tilde{R}_0 \tilde{d}.$$

We obtain a modification of Definition 9.1.

DEFINITION 9.3 *Let f, g_i be functions defined by (9.1), (9.2). Let $\mu_{\tilde{c}_j} : \mathbf{R} \to [0,1]$, $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and let $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, be membership functions of normal fuzzy quantities $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$, respectively. Moreover, let $\tilde{d} \in \mathcal{F}(\mathbf{R})$ be a fuzzy subset of the real line. Let $\tilde{R}_i$, $i \in \{0\} \cup \mathcal{M}$, be fuzzy relations on $\mathcal{F}(\mathbf{R})$ and let T be a t-norm, let A and A_G be aggregation operators.*

A fuzzy set $\tilde{X}^*$ with the membership function $\mu_{\tilde{X}^*}$ defined for all $x \in \mathbf{R}^n$ by

$$\mu_{\tilde{X}^*}(x) = A_G(\mu_{\tilde{R}_0}(\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n, \tilde{d}), \mu_{\tilde{X}}(x)), \qquad (9.18)$$

where $\mu_{\tilde{X}}(x)$ is the membership function of the feasible solution, is called the optimal solution *of FLP problem (9.5).*

For $\alpha \in (0,1]$ a vector $x \in [\tilde{X}^*]_\alpha$ is called the α-optimal solution *of FLP problem (9.5).*

A vector $x^* \in \mathbf{R}^n$ with the property

$$\mu_{\tilde{X}^*}(x^*) = \mathrm{Hgt}(\tilde{X}^*) \qquad (9.19)$$

is called the max-optimal solution.

Notice that the optimal solution of the FLP problem is a fuzzy set. On the other hand, the α-optimal solution is a vector belonging to the α-cut $[\tilde{X}^*]_\alpha$. Likewise, the max-optimal solution is an α-optimal solution with $\alpha = \mathrm{Hgt}(\tilde{X}^*)$. Notice that in view of Chapter 7 a max-optimal solution is in fact a max-A_G decision on $\mathbf{R}^n$.

In Definition 9.3, the t-norms T and the aggregation operators A and A_G have been used. The t-norm T has been used for extending arithmetic operations, the aggregation operator A for aggregating the individual constraints into the single feasible solution and A_G has been applied for aggregating the fuzzy set of the feasible solution with the fuzzy set of the objective $\tilde{X}_0$ defined by the membership function

$$\mu_{\tilde{X}_0}(x) = \mu_{\tilde{R}_0}(\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n, \tilde{d}), \qquad (9.20)$$

$x \in \mathbf{R}^n$. As a result, we have obtained the membership function of optimal solution $\tilde{X}^*$ defined for all $x \in \mathbf{R}^n$ by

$$\mu_{\tilde{X}^*}(x) = A_G(\mu_{\tilde{X}_0}(x), \mu_{\tilde{X}}(x)). \tag{9.21}$$

Since problem (9.5) is a maximization problem "the higher value is better", the membership function $\mu_{\tilde{d}}$ of $\tilde{d}$ is supposed to be increasing or non-decreasing. The fuzzy relation $\tilde{R}_0$ for comparing $\tilde{c}_1x_1\tilde{+}\cdots\tilde{+}\tilde{c}_nx_n$ and $\tilde{d}$ is supposed to be a fuzzy extension of $\geq$.

Formally, in Definitions 9.1 and 9.3, the concepts of feasible solution and optimal solution correspond to each other. Therefore, we can take advantage of some properties of feasible solutions studied in the preceding section.

We first observe that in case of crisp parameters c_j, a_{ij} and b_i, the set of all max-optimal solutions given by (9.19) coincides with the set of all optimal solutions of the crisp problem. We have the following theorem.

PROPOSITION 9.4 *Let c_j, a_{ij}, $b_i \in \mathbf{R}$ be crisp parameters of (9.5) for all $i \in \mathcal{M}$, $j \in \mathcal{N}$. Let $\tilde{d} \in \mathcal{F}(\mathbf{R})$ be a fuzzy goal with a strictly increasing membership function $\mu_{\tilde{d}}$. Let for $i \in \mathcal{M}$, $\tilde{R}_i$ be a fuzzy extension of relation "$\leq$" on $\mathbf{R}$, and $\tilde{R}_0 = \Psi^T(\geq)$ be a T-fuzzy extension of relation "$\geq$". Let T, A and A_G be t-norms.*

Then the set of all max-optimal solutions coincides with the set of all optimal solutions X^* of LP problem (9.3).

PROOF. Clearly, the feasible solution of (9.5) is crisp, i.e., $\mu_{\tilde{X}}(x) = \chi_X(x)$ for all $x \in \mathbf{R}^n$, where X is the set of all feasible solutions (9.4) of the crisp LP problem (9.3). Moreover, by (9.20), we obtain for crisp $c \in \mathbf{R}^n$

$$\mu_{\tilde{X}_0}(x) = \mu_{\tilde{R}_0}(f(x;c), \tilde{d}) = \mu_{\tilde{d}}(c_1x_1 + \cdots + c_nx_n).$$

Substituting (8.43) and (8.44) into (9.21) we obtain

$$\mu_{\tilde{X}^*}(x) = A_G(\mu_{\tilde{d}}(f(x,c)), \chi_X(x)) = \begin{cases} \mu_{\tilde{d}}(c_1x_1 + \cdots + c_nx_n) & \text{if } x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mu_{\tilde{d}}$ is strictly increasing, by (8.45) it follows that

$$\mu_{\tilde{X}^*}(x^*) = \mathrm{Hgt}(\tilde{X}^*)$$

if and only if

$$\mu_{\tilde{X}^*}(x^*) = \sup\{\mu_{\tilde{d}}(c_1x_1 + \cdots + c_nx_n) \mid x \in X\},$$

which is the desired result. ■

PROPOSITION 9.5 *Let $\tilde{c}'_j$, $\tilde{a}'_{ij}$ and $\tilde{b}'_i$ and $\tilde{c}''_j$, $\tilde{a}''_{ij}$ and $\tilde{b}''_i$ be two collections of fuzzy parameters of FLP problem (9.5), $i \in \mathcal{M}$, $j \in \mathcal{N}$. Let T, A, A_G be t-norms. Let $\tilde{R}_i = \Psi^T(R_i)$, $i \in \{0\} \cup \mathcal{M}$, be T-fuzzy extensions of valued relations R_i on $\mathbf{R}$.*

If $\tilde{X}^{*\prime}$ is the optimal solution of FLP problem (9.5) with the parameters $\tilde{c}'_j$, $\tilde{a}'_{ij}$ and $\tilde{b}'_i$, $\tilde{X}^{*\prime\prime}$ is the optimal solution of the FLP problem with the parameters $\tilde{c}''_j$, $\tilde{a}''_{ij}$ and $\tilde{b}''_i$ such that for all $i \in \mathcal{M}$, $j \in \mathcal{N}$,

$$\tilde{c}'_j \subset \tilde{c}''_j,\ \tilde{a}'_{ij} \subset \tilde{a}''_{ij} \text{ and } \tilde{b}'_i \subset \tilde{b}''_i,$$

then

$$\tilde{X}^{*\prime} \subset \tilde{X}^{*\prime\prime}.$$

PROOF. First we show that for the feasible solutions it holds $\tilde{X}' \subset \tilde{X}''$. Let $x \in \mathbf{R}^n$, $i \in \mathcal{M}$. Now we show that

$$\tilde{a}'_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}'_{in}x_n \subset \tilde{a}''_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}''_{in}x_n.$$

Indeed, by (6.15), for each $u \in \mathbf{R}$ we get

$$\begin{aligned}
&\mu_{\tilde{a}'_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}'_{in}x_n}(u) \\
&\qquad = \sup\{T(\mu_{\tilde{a}'_{i1}}(a_1), \ldots, \mu_{\tilde{a}'_{in}}(a_n)) \mid a_{i1}x_1 + \cdots + a_{in}x_n = u\} \\
&\qquad \leq \sup\{T(\mu_{\tilde{a}''_{i1}}(a_1), \ldots, \mu_{\tilde{a}''_{in}}(a_n)) \mid a_{i1}x_1 + \cdots + a_{in}x_n = u\} \\
&\qquad = \mu_{\tilde{a}''_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}''_{in}x_n}(u).
\end{aligned}$$

Now, as $\tilde{b}'_i \subset \tilde{b}''_i$, using monotonicity of T-fuzzy extension $\tilde{R}_i$ of R_i, we obtain

$$\mu_{\tilde{R}_i}(\tilde{a}'_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}'_{in}x_n, \tilde{b}'_i) \leq \mu_{\tilde{R}_i}(\tilde{a}''_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}''_{in}x_n, \tilde{b}''_i).$$

Then, applying again monotonicity of A in (9.7), we obtain $\tilde{X}' \subset \tilde{X}''$.

It remains to show that $\tilde{X}'_0 \subset \tilde{X}''_0$, where

$$\mu_{\tilde{X}'_0}(x) = \mu_{\tilde{R}_0}(\tilde{f}(x; \tilde{c}'), \tilde{d}),\ \mu_{\tilde{X}''_0}(x) = \mu_{\tilde{R}_0}(\tilde{f}(x; \tilde{c}''), \tilde{d}).$$

We show that $\tilde{f}(x; \tilde{c}') \subset \tilde{f}(x; \tilde{c}'')$. Indeed, since for all $j \in \mathcal{N}$, $\mu_{\tilde{c}'_j}(c) \leq \mu_{\tilde{c}''_j}(c)$ for all $c \in \mathbf{R}$, by (9.6), we obtain for all $u \in \mathbf{R}$

$$\begin{aligned}
&\mu_{\tilde{c}'_1x_1 \tilde{+} \cdots \tilde{+} \tilde{c}'_nx_n}(u) \\
&\qquad = \sup\{T(\mu_{\tilde{c}'_1}(c_1), \ldots, \mu_{\tilde{c}'_n}(c_n)) \mid c_1x_1 + \cdots + c_nx_n = u\} \\
&\qquad \leq \sup\{T(\mu_{\tilde{c}''_1}(c_1), \ldots, \mu_{\tilde{c}''_n}(c_n)) \mid c_1x_1 + \cdots + c_nx_n = u\} \\
&\qquad = \mu_{\tilde{c}''_1x_1 \tilde{+} \cdots \tilde{+} \tilde{c}''_nx_n}(u).
\end{aligned}$$

Again, using monotonicity of $\tilde{R}_0$, we have

$$\mu_{\tilde{R}_0}(\tilde{c}'_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}'_n x_n, \tilde{d}) \leq \mu_{\tilde{R}_0}(\tilde{c}''_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}''_n x_n, \tilde{d}).$$

Applying monotonicity of A_G in (9.21), we obtain $\tilde{X}^{*\prime} \subset \tilde{X}^{*\prime\prime}$. ■

Further on, we extend Proposition 9.2 to the case of the optimal solution of a FLP problem. For this purpose we add some notation. Given $\alpha \in (0, 1]$, $j \in \mathcal{N}$, let

$$\begin{aligned}
\underline{c}_j(\alpha) &= \inf\{c \mid c \in [\tilde{c}_j]_\alpha\}, \\
\overline{c}_j(\alpha) &= \sup\{c \mid c \in [\tilde{c}_j]_\alpha\}, \\
\underline{d}(\alpha) &= \inf\{d \mid d \in [\tilde{d}]_\alpha\}, \\
\overline{d}(\alpha) &= \sup\{d \mid d \in [\tilde{d}]_\alpha\}.
\end{aligned}$$

PROPOSITION 9.6 *Let, for all $i \in \mathcal{M}$, $j \in \mathcal{N}$, $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be compact, convex and normal fuzzy quantities, $\tilde{d} \in \mathcal{F}(\mathbf{R})$ be a fuzzy goal with the membership function $\mu_{\tilde{d}}$ satisfying the following conditions*

$$\begin{aligned}
&\mu_{\tilde{d}} \textit{ is upper semicontinuous,} \\
&\mu_{\tilde{d}} \textit{ is strictly increasing,} \\
&\lim_{t \to -\infty} \mu_{\tilde{d}}(t) = 0.
\end{aligned} \tag{9.22}$$

For $i \in \mathcal{M}$, let $\tilde{R}_i = \Psi^T(\leq)$ be the T-fuzzy extension of the binary relation $\leq$ on $\mathbf{R}$, $\tilde{R}_0 = \Psi^T(\geq)$ be the T-fuzzy extension of the binary relation $\geq$ on $\mathbf{R}$. Let $T = A = A_G = \min$. Let $\tilde{X}^*$ be an optimal solution of FLP problem (9.5) and $\alpha \in (0, 1)$.

A vector $x = (x_1, \ldots, x_n) \geq 0$ belongs to $[\tilde{X}^*]_\alpha$ if and only if

$$\sum_{j=1}^{n} \overline{c}_j(\alpha) x_j \;\geq\; \underline{d}(\alpha), \tag{9.23}$$

$$\sum_{j=1}^{n} \underline{a}_{ij}(\alpha) x_j \;\leq\; \overline{b}_i(\alpha), \quad i \in \mathcal{M}. \tag{9.24}$$

The proof is omitted since it is analogous to the proof of Proposition 9.2, part (i), with a simple modification that instead of compactness of $\tilde{d}$, we have assumptions (9.22).

For the sake of simplicity we confined ourselves in Proposition 9.6 only to the case of T-fuzzy extension of valued relations $\leq$ on $\mathbf{R}$. Evidently, similar results could be obtained for some other fuzzy extensions.

If the membership functions of the fuzzy parameters $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ can be formulated in an explicit form, e.g., similar to that of (6.16), (9.17), then we can

find an optimal solution with maximum height as the (crisp) optimal solution of some optimization problem.

PROPOSITION 9.7 *Consider FLP problem (9.5), where*

$$\mu_{\tilde{X}_0}(x) = \mu_{\tilde{R}_0}(\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n, \tilde{d})$$

and

$$\mu_{\tilde{X}_i}(x) = \mu_{\tilde{R}_i}(\tilde{a}_{i1} x_1 + \cdots + \tilde{a}_{in} x_n, \tilde{b}_i),$$

for each $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ *and every* $i \in \mathcal{M}$, *are the membership functions of the fuzzy objective and fuzzy constraints, respectively. Let* $T = A = A_G = \min$ *and let (9.22) hold for* $\tilde{d}$.

Then the vector $(t^*, x^*) \in \mathbf{R}^{n+1}$ *is an optimal solution of the optimization problem*

$$\begin{aligned} &\textit{maximize} \quad t \\ &\textit{subject to} \quad \mu_{\tilde{X}_i}(x) \geq t, \quad i \in \{0\} \cup \mathcal{M}, \\ &\qquad\qquad\quad x_j \geq 0, \quad j \in \mathcal{N} \end{aligned} \tag{9.25}$$

if and only if $x^* \in \mathbf{R}^n$ *is a max-optimal solution of FLP problem (9.5).*

PROOF. Let $(t^*, x^*) \in \mathbf{R}^{n+1}$ be an optimal solution of problem (9.25). By (9.18) and (9.19) we obtain

$$\mu_{\tilde{X}^*}(x^*) = \sup\{\min\{\mu_{\tilde{X}_0}(x), \mu_{\tilde{X}_i}(x)\} \mid x \in \mathbf{R}^n\} = \mathrm{Hgt}(\tilde{X}^*).$$

Hence, x^* is a max-optimal solution.

The proof of the converse statement is omitted. ∎

5. Extended Addition in FLP

In Theorems 9.2 and 9.6, formulae (9.13) and (9.24) hold, if the special case of $T = T_M$ is assumed. The t-norm T_M has been used not only for the T_M-fuzzy extensions of the binary relations on $\mathbf{R}$, but also for extending linear functions, that is, the objective function and constraints of the FLP problem.

In this section we shall investigate summations

$$\tilde{f}(x; \tilde{c}_1, \ldots, \tilde{c}_n) = \tilde{c}_1 x_1 \tilde{+}_T \cdots \tilde{+}_T \tilde{c}_n x_n, \tag{9.26}$$

and

$$\tilde{g}_i(x; \tilde{a}_{i1}, \ldots, \tilde{a}_{in}) = \tilde{a}_{i1} x_1 \tilde{+}_T \cdots \tilde{+}_T \tilde{a}_{in} x_n \tag{9.27}$$

for each $x \in \mathbf{R}^n$, where $\tilde{c}_j$, $\tilde{a}_{ij} \in \mathcal{F}(\mathbf{R})$, for all $i \in \mathcal{M}$, $j \in \mathcal{N}$. Formulae (9.26) and (9.27) are defined by (9.6) and (6.18), respectively, that is by using of the extension principle. Here $\tilde{+}_T$ denotes that the extended summation is performed by a t-norm T. Note that for arbitrary t-norms T, exact formulae

for (9.26) and (9.27) can be either complex or even inaccessible. However, in some special cases such formulae exist. Some of them are given below.

For the sake of brevity we deal only with (9.26), for (9.27) the results can be obtained analogously.

Let $F, G : (0, +\infty) \to [0, 1]$ be non-increasing left continuous functions. For $\gamma, \delta \in (0, +\infty)$, define functions $F_\gamma, G_\delta : (0, +\infty) \to [0, 1]$ by

$$F_\gamma(x) = F\left(\frac{x}{\gamma}\right), \quad G_\delta(x) = G\left(\frac{x}{\delta}\right),$$

where $x \in (0, +\infty)$. Let $l_j, r_j \in \mathbf{R}$ such that $l_j \le r_j$, let $\gamma_j, \delta_j \in (0, +\infty)$ and let

$$\tilde{c}_j = (l_j, r_j, F_{\gamma_j}, G_{\delta_j}), \quad j \in \mathcal{N},$$

be closed fuzzy intervals, with the membership functions given by

$$\mu_{\tilde{c}_j}(x) = \begin{cases} F_{\gamma_j}(l_j - x) & \text{if } x \in (-\infty, l_j), \\ 1 & \text{if } x \in [l_j, r_j], \\ G_{\delta_j}(x - r_j) & \text{if } x \in (r_j, +\infty), \end{cases} \tag{9.28}$$

see also Chapter 6. The following proposition shows that $\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n$ is closed fuzzy interval of the same type. The proof is omitted as it is a straightforward application of the extension principle. For the references, see [57].

PROPOSITION 9.8 *Let $\tilde{c}_j = (l_j, r_j, F_{\gamma_j}, G_{\delta_j})$, $j \in \mathcal{N}$, be closed fuzzy intervals with the membership functions given by (9.28). For $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $x_j \ge 0$ for all $j \in \mathcal{N}$, define I_x by*

$$I_x = \{j \mid x_j > 0, \ j \in \mathcal{N}\}.$$

Then

$$\tilde{c}_1 x_1 \tilde{+}_{T_M} \cdots \tilde{+}_{T_M} \tilde{c}_n x_n = (l, r, F_{l_M}, G_{r_M}), \tag{9.29}$$

$$\tilde{c}_1 x_1 \tilde{+}_{T_D} \cdots \tilde{+}_{T_D} \tilde{c}_n x_n = (l, r, F_{l_D}, G_{r_D}), \tag{9.30}$$

where T_M is the minimum t-norm, T_D is the drastic product and

$$l = \sum_{j \in I_x} l_j x_j, \qquad r = \sum_{j \in I_x} r_j x_j,$$

$$l_M = \sum_{j \in I_x} \frac{\gamma_j}{x_j}, \qquad r_M = \sum_{j \in I_x} \frac{\delta_j}{x_j},$$

$$l_D = \max\left\{\frac{\gamma_j}{x_j} \mid j \in I_x\right\}, \qquad r_D = \max\left\{\frac{\delta_j}{x_j} \mid j \in I_x\right\}.$$

If all $\tilde{c}_j$ are (L, R)-fuzzy intervals, then we can obtain an analogous and more specific result. Let $l_j, r_j \in \mathbf{R}$ with $l_j \leq r_j$, let $\gamma_j, \delta_j \in [0, +\infty)$ and let L, R be non-increasing, non-constant, upper-semicontinuous functions mapping the interval $(0, 1]$ into $[0, +\infty)$, Moreover, assume that $L(1) = R(1) = 0$, and define $L(0) = \lim_{x \to 0} L(x)$, $R(0) = \lim_{x \to 0} R(x)$.

For every $j \in \mathcal{N}$, let

$$\tilde{c}_j = (l_j, r_j, \gamma_j, \delta_j)_{LR}$$

be an (L, R)-fuzzy interval given by the membership function defined for each $x \in \mathbf{R}$ by

$$\mu_{\tilde{c}_j}(x) = \begin{cases} L^{(-1)}\left(\frac{l_j - x}{\gamma_j}\right) & \text{if } x \in (l_j - \gamma_j, l_j), \gamma_j > 0, \\ 1 & \text{if } x \in [l_j, r_j], \\ R^{(-1)}\left(\frac{x - r_j}{\delta_j}\right) & \text{if } x \in (r_j, r_j + \delta_j), \delta_j > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{9.31}$$

where $L^{(-1)}$, $R^{(-1)}$ are pseudo-inverse functions of L, R, respectively.

PROPOSITION 9.9 *Let $\tilde{c}_j = (l_j, r_j, \gamma_j, \delta_j)_{LR}$, $j \in \mathcal{N}$, be (L, R)-fuzzy intervals with the membership functions given by (9.31) and let $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $x_j \geq 0$ for all $j \in \mathcal{N}$. Then*

$$\tilde{c}_1 x_1 \tilde{+}_{T_M} \cdots \tilde{+}_{T_M} \tilde{c}_n x_n = (l, r, A_M, B_M)_{LR}, \tag{9.32}$$

$$\tilde{c}_1 x_1 \tilde{+}_{T_D} \cdots \tilde{+}_{T_D} \tilde{c}_n x_n = (l, r, A_D, B_D)_{LR}, \tag{9.33}$$

where T_M is the minimum t-norm, T_D is the drastic product and

$$l = \sum_{j \in \mathcal{N}} l_j x_j, \qquad r = \sum_{j \in \mathcal{N}} r_j x_j,$$

$$A_M = \sum_{j \in \mathcal{N}} \gamma_j x_j, \qquad B_M = \sum_{j \in \mathcal{N}} \delta_j x_j,$$

$$A_D = \max\{\gamma_j \mid j \in \mathcal{N}\}, \qquad B_D = \max\{\delta_j \mid j \in \mathcal{N}\}.$$

The results (9.30) and (9.33) in Proposition 9.8 and 9.9, respectively, can be extended as follows; see [57].

PROPOSITION 9.10 *Let T be a continuous Archimedian t-norm with an additive generator f. Let $F : (0, +\infty) \to [0, 1]$ be defined for each $x \in (0, +\infty)$ as*

$$F(x) = f^{(-1)}(x).$$

Let $\tilde{c}_j = (l_j, r_j, F_{\gamma_j}, F_{\delta_j})$, $j \in \mathcal{N}$, be closed fuzzy intervals with the membership functions given by (9.28) and let $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $x_j \geq 0$ for all $j \in \mathcal{N}$, $I_x = \{j \mid x_j > 0, j \in \mathcal{N}\}$. Then

$$\tilde{c}_1 x_1 \tilde{+}_T \cdots \tilde{+}_T \tilde{c}_n x_n = (l, r, F_{l_D}, F_{r_D}),$$

where

$$l = \sum_{j \in I_x} l_j x_j, \qquad r = \sum_{j \in I_x} r_j x_j,$$

$$l_D = \max\left\{\frac{\gamma_j}{x_j} \mid j \in I_x\right\}, \qquad r_D = \max\left\{\frac{\delta_j}{x_j} \mid j \in I_x\right\}.$$

Note that for a continuous Archimedean t-norm T and closed fuzzy intervals $\tilde{c}_j$ satisfying the assumptions of Proposition 9.10, we have

$$\tilde{c}_1 x_1 \tilde{+}_T \cdots \tilde{+}_T \tilde{c}_n x_n = \tilde{c}_1 x_1 \tilde{+}_{T_D} \cdots \tilde{+}_{T_D} \tilde{c}_n x_n,$$

which means that we obtain the same fuzzy linear function based on an arbitrary t-norm T' such that $T' \leq T$.

The following proposition generalizes several results concerning the addition of closed fuzzy intervals based on continuous Archimedean t-norms, see [57].

PROPOSITION 9.11 *Let T be a continuous Archimedean t-norm with an additive generator f. Let $K : [0, +\infty) \to [0, +\infty)$ be continuous convex function with $K(0) = 0$. Let $\alpha \in (0, +\infty)$ and*

$$F_\alpha(x) = f^{(-1)}\left(\alpha K\left(\frac{x}{\alpha}\right)\right)$$

for all $x \in [0, +\infty)$. Let $\tilde{c}_j = (l_j, r_j, F_{\gamma_j}, F_{\delta_j})$, $j \in \mathcal{N}$, be closed fuzzy intervals with the membership functions given by (9.28) and let $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $x_j \geq 0$ for all $j \in \mathcal{N}$, $I_x = \{j \mid x_j > 0, j \in \mathcal{N}\}$. Then

$$\tilde{c}_1 x_1 \tilde{+}_T \cdots \tilde{+}_T \tilde{c}_n x_n = (l, r, F_{l_K}, F_{r_K}),$$

where

$$l = \sum_{j \in I_x} l_j x_j, \qquad r = \sum_{j \in I_x} r_j x_j, \tag{9.34}$$

$$l_K = \sum_{j \in I_x} \frac{\gamma_j}{x_j}, \qquad r_K = \sum_{j \in I_x} \frac{\delta_j}{x_j}. \tag{9.35}$$

Two immediate consequences can be obtained from Proposition 9.10:

(i) The sum based on the product t-norm T_P of Gaussian fuzzy numbers, see Example 6.31, is again a Gaussian fuzzy number. Indeed, the additive generator f of the product t-norm T_P is given by $f(x) = -\log(x)$. Let $K(x) = x^2$. Then

$$F_\alpha(x) = f^{(-1)}\left(\alpha K\left(\frac{x}{\alpha}\right)\right) = e^{-\frac{x^2}{\alpha}}.$$

By Proposition 9.11 we obtain the required result.

(ii) The sum based on Yager t-norm T^Y_λ, see Example 4.17, of closed fuzzy intervals generated by the same K, is again a closed fuzzy interval of the same type. Observe that an additive generator f^Y_λ of the Yager t-norm T^Y_λ is given by $f^Y_\lambda(x) = (1-x)^\lambda$. For $\lambda \in (0, +\infty)$ we obtain

$$F_\alpha(x) = \max\left\{0, 1 - \frac{x}{\alpha^{\frac{\lambda-1}{\lambda}}}\right\}.$$

This means that each piecewise linear fuzzy number (l, r, γ, δ) can be written as

$$(l, r, \gamma, \delta) = \left(l, r, F_{\gamma^{\frac{\lambda}{\lambda-1}}}, F_{\delta^{\frac{\lambda}{\lambda-1}}}\right),$$

and the sum of piecewise linear fuzzy numbers $\tilde{c}_j = (l_j, r_j, \gamma_j, \delta_j)$, $j \in \mathcal{N}$, is again a piecewise linear fuzzy number

$$(l, r, \gamma, \delta),$$

where l and r are given by (9.34), and γ and δ are given as

$$\gamma = \sum_{j\in\mathcal{N}} \gamma_j^{\frac{\lambda}{\lambda-1}}, \quad \delta = \sum_{j\in\mathcal{N}} \delta_j^{\frac{\lambda}{\lambda-1}}.$$

The extensions can be obtained also for some other t-norms; see e.g. [57], [109].

An alternative approach based on centered fuzzy numbers will be mentioned later in this chapter; see also [61], [62].

6. Duality

In this section we generalize the well known concept of duality in LP for FLP problems. The results of this section, in a more general setting, can be found in [88]. We derive some weak and strong duality results which extend the known results for LP problems.

Consider the following FLP problem

$$\begin{array}{ll} \text{maximize} & \tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n \\ \text{subject to} & \tilde{a}_{i1} x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n \tilde{R} \tilde{b}_i, \quad i \in \mathcal{M}, \\ & x_j \geq 0, \quad j \in \mathcal{N}, \end{array} \tag{9.36}$$

where $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are normal fuzzy quantities with membership functions $\mu_{\tilde{c}_j} : \mathbf{R} \to [0, 1]$, $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0, 1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0, 1]$, $i \in \mathcal{M}$, $j \in \mathcal{N}$.

Let $\Psi : \mathcal{F}(\mathbf{R} \times \mathbf{R}) \to \mathcal{F}(\mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}))$ be the dual mapping to mapping $\Phi : \mathcal{F}(\mathbf{R} \times \mathbf{R}) \to \mathcal{F}(\mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}))$. Let R be a valued relation on $\mathbf{R}$ and

let $\tilde{R} = \Phi(R)$, ${}^*\tilde{R} = \Psi(R)$. Then by Definition 6.21 $\tilde{R}$ and ${}^*\tilde{R}$ are dual fuzzy relations.

Now, FLP problem (9.36) will be called the *primal FLP problem (P)*.

The *dual FLP problem (D)* is defined as

$$\begin{array}{ll} \text{minimize} & \tilde{b}_1 y_1 \tilde{+} \cdots \tilde{+} \tilde{b}_m y_m \\ \text{subject to} & \tilde{a}_{1j} y_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{mj} y_m {}^*\tilde{R}^{-1} \tilde{c}_j, \quad j \in \mathcal{N}, \\ & y_i \geq 0, \quad i \in \mathcal{M}. \end{array}$$

Here, ${}^*\tilde{R}^{-1}$ is the inverse fuzzy relation to ${}^*\tilde{R}$, that is, for all $A, B \in \mathcal{F}(\mathbf{R})$

$$\mu_{{}^*\tilde{R}^{-1}}(A, B) = \mu_{{}^*\tilde{R}}(B, A).$$

By Proposition 6.29, we can obtain a number of primal - dual pairs of FLP problems.

Let R be the usual binary operation $\leq$, let $T = \min$, $S = \max$. Let Ψ^T and Ψ_S be fuzzy extension mappings defined by (6.28) and (6.29), respectively. Since T is the dual t-norm to S, by Proposition 6.29, the mapping Ψ_S is dual to Ψ^T, hence by Definition 6.21, $\Psi_S(\leq)$ is the dual fuzzy relation to $\Psi^T(\leq)$. For the sake of simplicity, we denote

$$\tilde{\leq}^T \text{ by } \Psi^T(\leq), \quad \tilde{\leq}_S \text{ by } \Psi_S(\leq), \quad \text{and } \tilde{\geq}_S \text{ by } \Psi_S^{-1}(\leq).$$

We obtain the *primal - dual pair* of FLP problems with the fuzzy relation $\tilde{\leq}^T$ in (P) and the corresponding fuzzy relation $\tilde{\geq}_S$ in (D).

Now, consider the following pair of FLP problems

(P):

$$\begin{array}{ll} \text{maximize} & \tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n \\ \text{subject to} & \tilde{a}_{i1} x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n \tilde{\leq}^T \tilde{b}_i, \quad i \in \mathcal{M}, \\ & x_j \geq 0, \quad j \in \mathcal{N}. \end{array} \tag{9.37}$$

(D):

$$\begin{array}{ll} \text{minimize} & \tilde{b}_1 y_1 \tilde{+} \cdots \tilde{+} \tilde{b}_m y_m \\ \text{subject to} & \tilde{a}_{1j} y_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{mj} y_m \tilde{\geq}_S \tilde{c}_j, \quad j \in \mathcal{N}, \\ & y_i \geq 0, \quad i \in \mathcal{M}. \end{array} \tag{9.38}$$

Let the feasible solution of the primal FLP problem (P) be denoted by $\tilde{X}$, the feasible solution of the dual FLP problem (D) by $\tilde{Y}$. Clearly, $\tilde{X}$ is a fuzzy subset of $\mathbf{R}^n$, $\tilde{Y}$ is a fuzzy subset of $\mathbf{R}^m$.

Notice that in the crisp case, i.e., when the parameters $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are crisp real numbers, the relation $\tilde{\leq}^T$ on $\mathbf{R}$ coincides with $\leq$ and $\tilde{\geq}_S$ coincides with $\geq$, hence (P) and (D) is a primal - dual pair of LP problems in the usual sense.

In the following proposition we prove the weak form of the duality theorem for FLP problems.

PROPOSITION 9.12 *For all $i \in \mathcal{M}$ and $j \in \mathcal{N}$, let $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be compact, convex and normal fuzzy quantities. Let $A = T = \min$, $S = \max$. Let $\tilde{\leq}^T = \Psi^T(\leq)$ be the T-fuzzy extension of the binary relation $\leq$ on $\mathbf{R}$ defined by (6.28) and $\tilde{\geq}_S = \Psi_S^{-1}(\leq)$, where $\Psi_S(\leq)$ is the fuzzy extension of the relation $\leq$ defined by (6.29). Let $\tilde{X}$ be a feasible solution of FLP problem (9.37), $\tilde{Y}$ be a feasible solution of FLP problem (9.38) and $\alpha \in [0.5, 1)$.*

If a vector $x = (x_1, \ldots, x_n) \geq 0$ belongs to $[\tilde{X}]_\alpha$ and $y = (y_1, \ldots, y_m) \geq 0$ belongs to $[\tilde{Y}]_\alpha$, then

$$\sum_{j \in \mathcal{N}} \bar{c}_j(1-\alpha)x_j \leq \sum_{i \in \mathcal{M}} \bar{b}_i(1-\alpha)y_i. \tag{9.39}$$

PROOF. Let $x \in [\tilde{X}]_\alpha$ and $y \in [\tilde{Y}]_\alpha$, $x_j \geq 0, y_i \geq 0$ for all $i \in \mathcal{M}, j \in \mathcal{N}$. Then by Proposition 9.2 we obtain

$$\sum_{i=1}^{m} \underline{a}_{ij}(1-\alpha)y_i \geq \bar{c}_j(1-\alpha). \tag{9.40}$$

Since $\alpha \geq 0.5$, it follows that $1-\alpha \leq \alpha$, hence $[\tilde{X}]_\alpha \subset [\tilde{X}]_{1-\alpha}$. Again by Proposition 9.2 we obtain for all $i \in \mathcal{M}$

$$\sum_{j=1}^{n} \underline{a}_{ij}(1-\alpha)x_j \leq \bar{b}_i(1-\alpha). \tag{9.41}$$

Multiplying both sides of (9.40) and (9.41) by x_j and y_i, respectively, and summing up the results, we obtain

$$\sum_{j=1}^{n} \bar{c}_j(1-\alpha)x_j \leq \sum_{j=1}^{n}\sum_{i=1}^{m} \underline{a}_{ij}(1-\alpha)y_i x_j \leq \sum_{i=1}^{m} \bar{b}_i(1-\alpha)y_i,$$

which is the desired result. ■

Notice that in the crisp case, (9.39) is nothing else than the standard *weak duality*. Let us turn to the *strong duality*.

For this purpose, we assume the existence of exogenously given additional goals $\tilde{d} \in \mathcal{F}(\mathbf{R})$ and $\tilde{h} \in \mathcal{F}(\mathbf{R})$. The fuzzy goal $\tilde{d}$ is compared to fuzzy values $\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n$ of the objective function of the primal FLP problem (P) by fuzzy relation $\tilde{\geq}^T$. On the other hand, the fuzzy goal $\tilde{h}$ is compared to fuzzy

values $\tilde{b}_1 y_1 \tilde{+} \cdots \tilde{+} \tilde{b}_m y_m$ of the objective function of the dual FLP problem (D) by fuzzy relation $\tilde{\leq}_S$. In this way the fuzzy objectives are treated as constraints

$$\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n \tilde{\geq}^T \tilde{d},\ \tilde{b}_1 y_1 \tilde{+} \cdots \tilde{+} \tilde{b}_m y_m \tilde{\leq}_S \tilde{h}.$$

Let the optimal solution of the primal FLP problem (P), defined by Definition 9.3, be denoted by $\tilde{X}^*$, the optimal solution of the dual FLP problem (D), defined also by Definition 9.3, by $\tilde{Y}^*$. Clearly, $\tilde{X}^*$ is a fuzzy subset of $\mathbf{R}^n$, $\tilde{Y}^*$ is a fuzzy subset of $\mathbf{R}^m$, moreover, $\tilde{X}^* \subset \tilde{X}$ and $\tilde{Y}^* \subset \tilde{Y}$.

PROPOSITION 9.13 *For all $i \in \mathcal{M}$ and $j \in \mathcal{N}$, let $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be compact, convex and normal fuzzy quantities. Let $\tilde{d}, \tilde{h} \in \mathcal{F}(\mathbf{R})$ be fuzzy goals with the membership functions $\mu_{\tilde{d}}$ and $\mu_{\tilde{h}}$ satisfying the following conditions*

$$\begin{aligned} &\text{both } \mu_{\tilde{d}} \text{ and } \mu_{\tilde{h}} \text{ are upper semicontinuous,} \\ &\mu_{\tilde{d}} \text{ is strictly increasing, } \mu_{\tilde{h}} \text{ is strictly decreasing,} \\ &\lim_{t\to-\infty} \mu_{\tilde{d}}(t) = \lim_{t\to+\infty} \mu_{\tilde{h}}(t) = 0. \end{aligned} \tag{9.42}$$

Let $A = T = \min$, $S = \max$. Let $\tilde{\leq}^T = \Psi^T(\leq)$ be the T-fuzzy extension of the binary relation $\leq$ on $\mathbf{R}$ defined by (6.28) and $\tilde{\geq}_S = \Psi_S^{-1}(\leq)$, where $\Psi_S(\leq)$ is the fuzzy extension of the relation $\leq$ defined by (6.29). Let $\tilde{X}^*$ be an optimal solution of FLP problem (9.37), $\tilde{Y}^*$ be an optimal solution of FLP problem (9.38) and $\alpha \in (0,1)$.

If a vector $x^* = (x_1^*, \ldots, x_n^*) \geq 0$ belongs to $[\tilde{X}^*]_\alpha$, then there exists a vector $y^* = (y_1^*, \ldots, y_m^*) \geq 0$ which belongs to $[\tilde{Y}^*]_{1-\alpha}$, and

$$\sum_{j\in\mathcal{N}} \bar{c}_j(\alpha) x_j^* = \sum_{i\in\mathcal{M}} \bar{b}_i(\alpha) y_i^*. \tag{9.43}$$

PROOF. Let $x^* = (x_1^*, \ldots, x_n^*) \geq 0$, $x^* \in [\tilde{X}^*]_\alpha$. By Proposition 9.6

$$\sum_{j=1}^{n} \bar{c}_j(\alpha) x_j^* \geq \underline{d}(\alpha), \tag{9.44}$$

$$\sum_{j=1}^{n} \underline{a}_{ij}(\alpha) x_j^* \leq \bar{b}_i(\alpha), \quad i \in \mathcal{M}. \tag{9.45}$$

Consider the following LP problem:

(P1)

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_{j=1}^{n} \bar{c}_j(\alpha) x_j \\ \text{subject to} \quad & \textstyle\sum_{j=1}^{n} \underline{a}_{ij}(\alpha) x_j \leq \bar{b}_i(\alpha), \quad i \in \mathcal{M}, \\ & x_j \geq 0, \quad j \in \mathcal{N}. \end{aligned}$$

By conditions (9.42) concerning the fuzzy goal $\tilde{d}$, the system of inequalities (9.44) and (9.45) is satisfied if and only if x^* is the optimal solution of (P1). By the standard strong duality theorem for LP, there exists $y^* \in \mathbf{R}^*$ being an optimal solution of the dual problem

(D1)

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} \bar{b}_i(\alpha) y_i \\ \text{subject to} & \sum_{i=1}^{m} \underline{a}_{ij}(\alpha) y_i \geq \bar{c}_j(\alpha), \quad j \in \mathcal{N}, \\ & y_i \geq 0, \quad i \in \mathcal{M}. \end{array}$$

such that (9.43) holds.

It remains only to prove that $y^* \in [\tilde{Y}^*]_{1-\alpha}$. This fact follows, however, from conditions (9.42) concerning the fuzzy goal $\tilde{h}$, and from (9.14). ■

Notice that in the crisp case, (9.43) is the standard strong duality result for LP.

7. Special Models of FLP

Several models of FLP problem known from the literature are investigated in this section.

7.1. Interval Linear Programming

In this subsection we apply the previous results to a special case of the FLP problem - interval linear programming problem. By *interval linear programming problem (ILP)* we understand the following FLP problem

$$\begin{array}{ll} \text{maximize} & \tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n \\ \text{subject to} & \tilde{a}_{i1} x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n \tilde{R} \tilde{b}_i, \quad i \in \mathcal{M}, \\ & x_j \geq 0, \quad j \in \mathcal{N}, \end{array} \tag{9.46}$$

where the parameters $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are considered to be compact intervals in $\mathbf{R}$, i.e., $\tilde{c}_j = [\underline{c}_j, \bar{c}_j]$, $\tilde{a}_{ij} = [\underline{a}_{ij}, \bar{a}_{ij}]$ and $\tilde{b}_i = [\underline{b}_i, \bar{b}_i]$, where $\underline{c}_j$, $\bar{c}_j$, $\underline{a}_{ij}$, $\bar{a}_{ij}$ and $\underline{b}_i$, $\bar{b}_i$ are lower and upper bounds of the corresponding intervals, respectively. The membership functions of $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are the characteristic functions of the intervals, i.e., $\chi_{[\underline{c}_j, \bar{c}_j]} : \mathbf{R} \to [0,1]$, $\chi_{[\underline{a}_{ij}, \bar{a}_{ij}]} : \mathbf{R} \to [0,1]$ and $\chi_{[\underline{b}_i, \bar{b}_i]} : \mathbf{R} \to [0,1]$, $i \in \mathcal{M}$, $j \in \mathcal{N}$. The fuzzy relation $\tilde{R}$ is considered to be the fuzzy extension of a valued relation R on $\mathbf{R}$. We assume that R is the usual binary relation $\leq$, and $A = T = \min$, $S = \max$. Then, for fuzzy relation $\tilde{R}$, we have

$$\tilde{R} \in \left\{ \tilde{\leq}^T, \tilde{\leq}_S, \tilde{\leq}^{T,S}, \tilde{\leq}_{T,S}, \tilde{\leq}^{S,T}, \tilde{\leq}_{S,T} \right\}.$$

Then by Proposition 9.2 we obtain 6 types of feasible solutions of ILP problem (9.46):

(i)

$$X_{\tilde{\leq}^T} = \left\{ x \in \mathbf{R}^n \;\middle|\; \sum_{j=1}^{n} \underline{a}_{ij} x_j \leq \bar{b}_i,\ x_j \geq 0,\ j \in \mathcal{N} \right\}. \tag{9.47}$$

(ii)

$$X_{\tilde{\leq}_S} = \left\{ x \in \mathbf{R}^n \;\middle|\; \sum_{j=1}^{n} \bar{a}_{ij} x_j \leq \underline{b}_i,\ x_j \geq 0,\ j \in \mathcal{N} \right\}. \tag{9.48}$$

(iii)

$$X_{\tilde{\leq}^{T,S}} = X_{\tilde{\leq}_{T,S}} = \left\{ x \in \mathbf{R}^n \;\middle|\; \sum_{j=1}^{n} \bar{a}_{ij} x_j \leq \bar{b}_i,\ x_j \geq 0,\ j \in \mathcal{N} \right\}. \tag{9.49}$$

(iv)

$$X_{\tilde{\leq}^{S,T}} = X_{\tilde{\leq}_{S,T}} = \left\{ x \in \mathbf{R}^n \;\middle|\; \sum_{j=1}^{n} \underline{a}_{ij} x_j \leq \underline{b}_i,\ x_j \geq 0,\ j \in \mathcal{N} \right\}. \tag{9.50}$$

Clearly, feasible solutions (9.47) - (9.50) are crisp subsets of $\mathbf{R}^n$, moreover, they all are polyhedral.

In order to find an optimal solution of ILP problem (9.46), we consider a fuzzy goal $\tilde{d} \in \mathcal{F}(\mathbf{R})$ and $\tilde{R}_0$, a fuzzy extension of the usual binary relation $\geq$ for comparing the objective with the fuzzy goal.

In the following proposition we show that if the feasible solution of ILP problem is crisp then its max-optimal solution is the same as the set of all optimal solution of the problem of maximizing a particular crisp objective over the set of feasible solutions.

PROPOSITION 9.14 *Let X be a crisp feasible solution of ILP problem (9.46). Let $\tilde{d} \in \mathcal{F}(\mathbf{R})$ be a fuzzy goal with the membership function $\mu_{\tilde{d}}$ satisfying conditions (9.22). Let $A_G = A = T = \min$, $S = \max$.*

(i) *If $\tilde{R}_0$ is $\tilde{\geq}^T$, then the set of all max-optimal solutions of ILP problem (9.46) coincides with the set of all optimal solution of the problem*

$$\begin{array}{ll} \textit{maximize} & \sum_{j=1}^{n} \bar{c}_j x_j \\ \textit{subject to} & x \in X. \end{array} \tag{9.51}$$

(ii) *If $\tilde{R}_0$ is $\tilde{\geq}_S$, then the set of all max-optimal solutions of ILP problem (9.46) coincides with the set of all optimal solution of the problem*

$$\begin{array}{ll} \textit{maximize} & \sum_{j=1}^{n} \underline{c}_j x_j \\ \textit{subject to} & x \in X. \end{array}$$

PROOF. (i) Let $x \in X$ be a max-optimal solution of ILP problem (9.46), $\underline{c} = \sum_{j=1}^{n} \underline{c}_j x_j$, $\bar{c} = \sum_{j=1}^{n} \bar{c}_j x_j$. By our assumptions, (6.29) and (9.22) give

$$\begin{aligned} &\mu_{\tilde{\geq}^T}(\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n, \tilde{d}) \\ &\quad = \sup\{\min\{\mu_{\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n}(u), \mu_{\tilde{d}}(v)\} \mid u \geq v\} \\ &\quad = \sup\{\min\{\chi_{[\underline{c},\bar{c}]}(u), \mu_{\tilde{d}}(v)\} \mid u \geq v\} \\ &\quad = \mu_{\tilde{d}}\left(\sum_{j=1}^{n} \bar{c}_j x_j\right). \end{aligned}$$

Hence, x is an optimal solution of (9.51). Conversely, if $x \in X$ is an optimal solution of (9.51), then by Definition 9.3 and by (9.22), x is a max-optimal solution of problem (9.46).

(ii) Analogously to the proof of (i), we have

$$\begin{aligned} &\mu_{\tilde{\geq}_S}(\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n, \tilde{d}) \\ &\quad = \inf\{\max\{1 - \mu_{\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n}(u), 1 - \mu_{\tilde{d}}(v)\} \mid u \geq v\} \\ &\quad = \inf\{1 - \min\{\chi_{[\underline{c},\bar{c}]}(u), \mu_{\tilde{d}}(v)\} \mid u \leq v\} \\ &\quad = \mu_{\tilde{d}}\left(\sum_{j=1}^{n} \underline{c}_j x_j\right). \end{aligned}$$

By the same arguments as in (i) we conclude the proof. ∎

We close this section with several observations about duality of ILP problems.

Let the primal ILP problem (P) be problem (9.46) with $\tilde{R} = \tilde{\leq}^T$, i.e., (9.37). Then the dual ILP problem (D) is (9.38). Clearly, the feasible solution $X_{\tilde{\leq}^T}$ of (P) is defined by (9.47) and the feasible solution $Y_{\tilde{\geq}_S}$ of the dual problem (D) can be derived from (9.48) as

$$Y_{\tilde{\geq}_S} = \left\{ y \in \mathbf{R}^m \;\middle|\; \sum_{i=1}^{m} \underline{a}_{ij} y_i \geq \bar{c}_j,\, y_i \geq 0,\, i \in \mathcal{M} \right\}.$$

Notice that the problems

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{n} \bar{c}_j x_j \\ \text{subject to} & x \in X_{\tilde{\leq}^T} \end{array}$$

and

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} \underline{b}_i y_i \\ \text{subject to} & y \in Y_{\tilde{\geq}_S} \end{array}$$

are dual to each other in the usual (crisp) sense if and only if $\underline{c}_j = \overline{c}_j$ and $\underline{b}_i = \overline{b}_i$ for all $i \in \mathcal{M}$ and $j \in \mathcal{N}$.

For ILP problems our results correspond to that of [119], [128], [129].

7.2. Flexible Linear Programming

The term *flexible linear programming* is referred to the approach to LP problems allowing for a kind of flexibility of the objective function and constraints in standard LP problem (9.3), that is

$$\begin{array}{ll} \text{maximize} & c_1x_1 + \cdots + c_nx_n \\ \text{subject to} & a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i, \quad i \in \mathcal{M}, \\ & x_j \geq 0, \quad j \in \mathcal{N}. \end{array} \tag{9.52}$$

see [138], [109]. The values of parameters c_j, a_{ij} and b_i in (9.52) are supposed to be uncertain, not confident, etc. Nonnegative values p_i, $i \in \{0\} \cup \mathcal{M}$, of admissible violations of the objective and constraints are (subjectively) chosen and introduced to the original model (9.52).

For the objective function, an aspiration value $d_0 \in \mathbf{R}$ is also (subjectively) determined such that if the value of the objective function is greater or equal to d_0, then the decision maker (DM) is fully satisfied. On the other hand, if the objective function attains a value smaller than $d_0 - p_0$, then (DM) is fully dissatisfied. Within the interval $(d_0 - p_0, d_0)$, the satisfaction of DM increases linearly from 0 to 1. By these considerations a membership function $\mu_{\tilde{d}}$ of the fuzzy goal $\tilde{d}$ is defined as follows

$$\mu_{\tilde{d}}(t) = \begin{cases} 1 & \text{if } t \geq d_0, \\ 1 + \frac{t-d_0}{p_0} & \text{if } d_0 - p_0 \leq t < d_0, \\ 0 & \text{otherwise.} \end{cases} \tag{9.53}$$

Similarly, let for the ith constraint function of (9.52), $i \in \mathcal{M}$, a right hand side $b_i \in \mathbf{R}$ is known such that if the left hand side attains this value, or if it is below it, then the decision maker (DM) is fully satisfied. On the other hand, if the objective function attains its value greater than $b_i + p_i$, then (DM) is fully dissatisfied. Within the interval $(b_i, b_i + p_i)$, the satisfaction of DM decreases linearly from 1 to 0. By these considerations the membership function $\mu_{\tilde{b}_i}$ of the fuzzy right hand side $\tilde{b}_i$ is defined as

$$\mu_{\tilde{b}_i}(t) = \begin{cases} 1 & \text{if } t \leq b_i, \\ 1 - \frac{t-b_i}{p_i} & \text{if } b_i \leq t < b_i + p_i, \\ 0 & \text{otherwise.} \end{cases} \tag{9.54}$$

The relationship between the objective function and constraints in the flexible LP problem is considered as fully symmetric; i.e., there is no longer a difference between the former and the latter. "Maximization" is then understood as finding a vector $x \in \mathbf{R}^n$ such that the membership grade of the intersection of all fuzzy sets (9.53) and (9.54) is maximized. In other words, we have to solve the following optimization problem:

$$\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & \mu_{\tilde{d}}\left(\sum_{j\in\mathcal{N}} c_j x_j\right) \geq \lambda, \\ & \mu_{\tilde{b}_i}\left(\sum_{j\in\mathcal{N}} a_{ij} x_j\right) \geq \lambda, \quad i \in \mathcal{M}, \\ & 0 \leq \lambda \leq 1, \\ & x_j \geq 0, \quad j \in \mathcal{N}. \end{array} \tag{9.55}$$

Problem (9.55) can easily be transformed to the equivalent LP problem:

$$\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & \sum_{j\in\mathcal{N}} c_j x_j \geq d_0 + \lambda p_0, \\ & \sum_{j\in\mathcal{N}} a_{ij} x_j \leq b_i + (1-\lambda) p_i, \quad i \in \mathcal{M}, \\ & 0 \leq \lambda \leq 1, \\ & x_j \geq 0, \quad j \in \mathcal{N}. \end{array} \tag{9.56}$$

In Section 2, we introduced FLP problem (9.5). Now, consider the following, a more specific FLP problem:

$$\begin{array}{ll} \text{maximize} & c_1 x_1 + \cdots + c_n x_n \\ \text{subject to} & a_{i1} x_1 + \cdots + a_{in} x_n \tilde{\leq}^T \tilde{b}_i, \quad i \in \mathcal{M}, \\ & x_j \geq 0, \quad j \in \mathcal{N}. \end{array} \tag{9.57}$$

where c_j, a_{ij} and b_i are the same as above, that is, crisp numbers, whereas $\tilde{d}$ and $\tilde{b}_i$ are fuzzy quantities defined by (9.53) and (9.54). Moreover, $\tilde{\leq}^T$ is a T-fuzzy extension of the usual inequality relation $\leq$, with $T = \min$. It turns out that the vector $x \in \mathbf{R}^n$ is an optimal solution of flexible LP problem (9.56) if and only if it is a max-optimal solution of FLP problem (9.57). This statement follows directly from Proposition 9.7.

Notice that piecewise linear membership functions (9.53) and (9.54) can be replaced by more general nondecreasing and nonincreasing functions, respectively. In general, problem (9.55) cannot be equivalently transformed to the LP problem (9.56). Such transformation is, however, possible, e.g., if all membership functions are generated by the same strictly monotone function.

7.3. FLP Problems with Interactive Fuzzy Parameters

In this subsection we shall deal with a fuzzy linear programming problem with the parameters being interactive fuzzy quantities introduced in Chapter 6.9.

Let f, g_i be functions defined by (9.1), (9.2), i.e.,

$$\begin{aligned} f(x; c_1, \ldots, c_n) &= c_1x_1 + \cdots + c_nx_n, \\ g_i(x; a_{i1}, \ldots, a_{in}) &= a_{i1}x_1 + \cdots + a_{in}x_n, \quad i \in \mathcal{M}. \end{aligned}$$

The parameters c_j, a_{ij} and b_i will be considered as normal fuzzy quantities, that is normal fuzzy subsets of the Euclidean space $\mathbf{R}$. Let $\mu_{\tilde{c}_j} : \mathbf{R} \to [0,1]$, $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, be membership functions of the fuzzy parameters $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$, respectively.

Let $\tilde{R}_i$, $i \in \mathcal{M}$, be fuzzy relations on $\mathcal{F}(\mathbf{R})$. Again, we have an exogenously given fuzzy goal $\tilde{d} \in \mathcal{F}(\mathbf{R})$ and associated fuzzy relation $\tilde{R}_0$ on $\mathbf{R}$. Moreover, let $D_i = (d_{i1i}, d_{i2}, \ldots, d_{im})$ be nonsingular $n \times n$ matrices - obliquity matrices, where all $d_{ij} \in \mathbf{R}^n$ are columns of matrices D_i, $i = \{0\} \cup \mathcal{M}$.

The *fuzzy linear programming problem with interactive parameters* (FLP problem with IP) associated with LP problem (9.3) is formulated as

$$\begin{array}{ll} \text{maximize} & \tilde{c}_1x_1 \tilde{+}^{D_0} \cdots \tilde{+}^{D_0} \tilde{c}_nx_n \\ \text{subject to} & \tilde{a}_{i1}x_1 \tilde{+}^{D_i} \cdots \tilde{+}^{D_i} \tilde{a}_{in}x_n \tilde{R}_i \tilde{b}_i, \quad i \in \mathcal{M}, \\ & x_j \geq 0, \quad j \in \mathcal{N}. \end{array} \tag{9.58}$$

Let us clarify the elements of (9.58).

The objective function values and the left hand sides of the constraints of (9.58) have been obtained by the extension principle (6.15) as follows. By (6.40) we obtain

$$\mu_{\tilde{a}_i}(a) = T(\mu_{\tilde{a}_{i1}}(\langle d_{i1}, a\rangle), \mu_{\tilde{a}_{i2}}(\langle d_{i2}, a\rangle), \ldots, \mu_{\tilde{a}_{in}}(\langle d_{in}, a\rangle)).$$

A membership function of $\tilde{g}_i(x; \tilde{a}_i)$ is defined for each $t \in \mathbf{R}$ by

$$\mu_{\tilde{g}_i}(t) = \begin{cases} \sup\left\{ \mu_{\tilde{a}_i}(a) \;\middle|\; \begin{array}{l} a = (a_1, \ldots, a_n) \in \mathbf{R}^n, \\ a_1x_1 + \cdots + a_nx_n = t \end{array} \right\} & \text{if } g_i^{-1}(x;t) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $g_i^{-1}(x;t) = \{(a_1, \ldots, a_n) \in \mathbf{R}^n \mid a_1x_1 + \cdots + a_nx_n = t\}$. Here, the fuzzy set $\tilde{g}_i(x; \tilde{a}_i)$ is denoted as $\tilde{a}_{i1}x_1 \tilde{+}^{D_i} \cdots \tilde{+}^{D_i} \tilde{a}_{in}x_n$, i.e.,

$$\tilde{g}_i(x; \tilde{a}_i) = \tilde{a}_{i1}x_1 \tilde{+}^{D_i} \cdots \tilde{+}^{D_i} \tilde{a}_{in}x_n \tag{9.59}$$

for every $i \in \mathcal{M}$ and for each $x \in \mathbf{R}^n$.

Also, for given interactive $\tilde{c}_1, \ldots, \tilde{c}_n \in \mathcal{F}(\mathbf{R})$, by (6.40) we obtain

$$\mu_{\tilde{c}}(c) = T(\mu_{\tilde{c}_1}(\langle d_{01}, c\rangle), \mu_{\tilde{c}_2}(\langle d_{02}, c\rangle), \ldots, \mu_{\tilde{c}_n}(\langle d_{0n}, c\rangle)). \tag{9.60}$$

A membership function of $\tilde{f}(x; \tilde{c})$ is defined for each $t \in \mathbf{R}$ by

$$\mu_{\tilde{f}}(t) = \begin{cases} \sup\left\{\mu_{\tilde{c}}(c) \;\middle|\; \begin{array}{l} c = (c_1, \ldots, c_n) \in \mathbf{R}^n, \\ c_1x_1 + \cdots + c_nx_n = t \end{array}\right\} & \text{if } f^{-1}(x;t) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{9.61}$$

where $f^{-1}(x;t) = \{(c_1, \ldots, c_n) \in \mathbf{R}^n \mid c_1x_1 + \cdots + c_nx_n = t\}$. Here, the fuzzy set $\tilde{f}(x; \tilde{c})$ is denoted as $\tilde{c}_1x_1 \tilde{+}^{D_0} \cdots \tilde{+}^{D_0} \tilde{c}_nx_n$, i.e.,

$$\tilde{f}(x; \tilde{c}) = \tilde{c}_1x_1 \tilde{+}^{D_0} \cdots \tilde{+}^{D_0} \tilde{c}_nx_n \tag{9.62}$$

for each $x \in \mathbf{R}^n$.

The treatment of FLP problem (9.58) is analogous to that of (9.58). The following proposition demonstrates how the α-cuts of (9.59) and (9.62) can be calculated.

Let D_0 be a non-singular obliquity matrix, denote $D_0^{-1} = \{d^*_{ij}\}_{i,j=1}^n$. For $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ we denote

$$I_x^+ = \{i \in \mathcal{N} \mid x_i \geq 0\},\ I_x^- = \{i \in \mathcal{N} \mid x_i < 0\}$$

and for all $i \in \mathcal{N}$

$$x_i^* = \sum_{j=1}^{n} d^*_{ij} x_j. \tag{9.63}$$

Given $\alpha \in (0, 1], j \in \mathcal{N}$, let

$$\begin{aligned} \underline{c}_j(\alpha) &= \inf\{c \mid c \in [\tilde{c}_j]_\alpha\}, \\ \overline{c}_j(\alpha) &= \sup\{c \mid c \in [\tilde{c}_j]_\alpha\}. \end{aligned}$$

PROPOSITION 9.15 *Let $\tilde{c}_1, \ldots, \tilde{c}_n \in \mathcal{F}_I(\mathbf{R})$ be compact interactive fuzzy intervals with an obliquity matrix D_0, $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$. Let T be a continuous t-norm and $\tilde{f}(x; \tilde{c}) = \tilde{c}_1x_1 \tilde{+}^{D_0} \cdots \tilde{+}^{D_0} \tilde{c}_nx_n$ be defined by (9.61), $\alpha \in (0, 1]$. Then*

$$\left[\tilde{f}(x; \tilde{c})\right]_\alpha = \left[\sum_{j \in I^+_{x^*}} \underline{c}_j(\alpha)x_j^* + \sum_{j \in I^-_{x^*}} \overline{c}_j(\alpha)x_j^*, \sum_{j \in I^+_{x^*}} \overline{c}_j(\alpha)x_j^* + \sum_{j \in I^-_{x^*}} \underline{c}_j(\alpha)x_j^*\right].$$

PROOF. Observe that $[\tilde{c}_j]_\alpha = [\underline{c}_j(\alpha), \overline{c}_j(\alpha)]$. The proof follows directly from (9.60), (9.61), (9.63) and Theorem 6.51. ∎

Analogical result can be formulated and proved for interactive fuzzy parameters in the constraints of (9.58), i.e., if $\tilde{a}_{i1}, \ldots, \tilde{a}_{in} \in \mathcal{F}_I(\mathbf{R})$ are compact interactive fuzzy intervals with an obliquity matrix D_i, $i \in \mathcal{M}$. Then we can take advantage of Theorem 9.2.

The practical difficulty of FLP with interactive parameters is that the membership functions of interactive parameters $\tilde{c}_j$, $\tilde{a}_{ij}$ are not observable. Instead, marginal fuzzy parameters can be measured or estimated. The problem of a unique representation of interactive fuzzy parameters by their marginals has been solved in [47] and [94].

7.4. FLP Problems with Centered Parameters

Interesting FLP models can be obtained if the parameters of the FLP problem are supposed to be centered fuzzy numbers called $\mathcal{B}$-fuzzy intervals; see Definition 6.34 in Chapter 6, or [60].

Let $\mathcal{B}$ be a basis of generators ordered by inclusion $\subset$. Let $\leq_\mathcal{B}$ be a partial ordering on the set $\mathcal{F}_\mathcal{B}(\mathbf{R})$ of all $\mathcal{B}$-fuzzy intervals on $\mathbf{R}$, defined by (6.37) in Definition 6.34. Obviously, if $\mathcal{B}$ is linearly ordered by $\subset$, then $\mathcal{F}_\mathcal{B}(\mathbf{R})$ is linearly ordered by $\leq_\mathcal{B}$. By Definition 6.34, each $\tilde{c} \in \mathcal{F}_\mathcal{B}(\mathbf{R})$ can be uniquely represented by a pair (c, μ), where $c \in \mathbf{R}$ and $\mu \in \mathcal{B}$ such that

$$\mu_{\tilde{c}}(t) = \mu(c - t),$$

we can write $\tilde{c} = (c, \mu)$.

Let $\circ$ be either addition $+$ or multiplication $\cdot$ arithmetic operations on $\mathbf{R}$ and $\star$ be either min or max operations on $\mathcal{B}$. Let us introduce on $\mathcal{F}_\mathcal{B}(\mathbf{R})$ the following four operations:

$$(a, f) \circ^{(\star)} (b, g) = (a \circ b, f \star g) \tag{9.64}$$

for all $(a, f), (b, g) \in \mathcal{F}_\mathcal{B}(\mathbf{R})$. It can easily be proved that the pairs of operations $(+^{(\min)}, \cdot^{(\min)})$, $(+^{(\min)}, \cdot^{(\max)})$, $(+^{(\max)}, \cdot^{(\min)})$, and $(+^{(\max)}, \cdot^{(\max)})$, are distributive. For more properties of these operations, see [62].

Now, let $\tilde{c}_j = (c_j, f_j)$, $\tilde{a}_{ij} = (a_{ij}, g_{ij})$, $\tilde{b}_i = (b_i, h_i)$, $\tilde{c}_j, \tilde{a}_{ij}, \tilde{b}_i \in \mathcal{F}_\mathcal{B}(\mathbf{R})$ be $\mathcal{B}$-fuzzy intervals, $i \in \mathcal{M}$, $j \in \mathcal{N}$. Let $\diamond$ and $\star$ be either min or max operations on $\mathcal{B}$. Consider the following optimization problem:

$$\begin{aligned}
&\text{maximize} && \tilde{c}_1 \cdot^{(\diamond)} \tilde{x}_1 +^{(\star)} \cdots +^{(\star)} \tilde{c}_n \cdot^{(\diamond)} \tilde{x}_n \\
&\text{subject to} && \tilde{a}_{i1} \cdot^{(\diamond)} \tilde{x}_1 +^{(\star)} \cdots +^{(\star)} \tilde{a}_{in} \cdot^{(\diamond)} \tilde{x}_n \leq_\mathcal{B} \tilde{b}_i, \quad i \in \mathcal{M}, \\
& && \tilde{x}_j \geq_\mathcal{B} \tilde{0}, \quad j \in \mathcal{N}.
\end{aligned} \tag{9.65}$$

Here, maximization is performed with respect to the ordering $\leq_{\mathcal{B}}$. Moreover, $\tilde{x}_j = (x_j, \xi_j)$, where $x_j \in \mathbf{R}$ and $\xi_j \in \mathcal{B}$, $\tilde{0} = (0, \chi_{\{0\}})$. The constraints $\tilde{x}_j \geq_{\mathcal{B}} \tilde{0}, j \in \mathcal{N}$, are equivalent to $x_j \geq 0, j \in \mathcal{N}$. Comparing to the previous approach, we consider a different concept of feasible and optimal solution.

A *feasible solution* of the optimization problem (9.65) is a vector

$$(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \in \mathcal{F}_{\mathcal{B}}(\mathbf{R}) \times \mathcal{F}_{\mathcal{B}}(\mathbf{R}) \times \cdots \times \mathcal{F}_{\mathcal{B}}(\mathbf{R}),$$

satisfying the constraints

$$\begin{aligned} \tilde{a}_{i1} \cdot^{(\diamond)} \tilde{x}_1 +^{(\star)} \cdots +^{(\star)} \tilde{a}_{in} \cdot^{(\diamond)} \tilde{x}_n &\leq_{\mathcal{B}} \tilde{b}_i, \quad i \in \mathcal{M}, \\ \tilde{x}_j &\geq_{\mathcal{B}} \tilde{0}, \quad j \in \mathcal{N}. \end{aligned}$$

The set of all feasible solutions of (9.65) is denoted by $X_{\mathcal{B}}$.

An *optimal solution* of the optimization problem (9.65) is a vector

$$(\tilde{x}_1^*, \tilde{x}_2^*, \cdots, \tilde{x}_n^*) \in \mathcal{F}_{\mathcal{B}}(\mathbf{R}) \times \mathcal{F}_{\mathcal{B}}(\mathbf{R}) \times \cdots \times \mathcal{F}_{\mathcal{B}}(\mathbf{R})$$

such that

$$\tilde{z}^* = \tilde{c}_1 \cdot^{(\diamond)} \tilde{x}_1^* +^{(\star)} \cdots +^{(\star)} \tilde{c}_n \cdot^{(\diamond)} \tilde{x}_n^*$$

is the maximal element of the set

$$X_{\mathcal{B}}^* = \{\tilde{z} \mid \tilde{z} = \tilde{c}_1 \cdot^{(\diamond)} \tilde{x}_1 +^{(\star)} \cdots +^{(\star)} \tilde{c}_n \cdot^{(\diamond)} \tilde{x}_n, (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \in X_{\mathcal{B}}\}.$$

Notice that for each of four possible combinations of min and max in the operations $\cdot^{(\diamond)}$ and $+^{(\star)}$, (9.65) defines in fact an individual optimization problem.

PROPOSITION 9.16 *Let $\mathcal{B}$ be a linearly ordered basis of generators. Let $(\tilde{x}_1^*, \tilde{x}_2^*, \ldots, \tilde{x}_n^*) \in \mathcal{F}_{\mathcal{B}}(\mathbf{R})^n$ be an optimal solution of (9.65), where $\tilde{x}_j^* = (x_j^*, \xi_j^*)$, $j \in \mathcal{N}$. Then the vector $x^* = (x_1^*, \ldots, x_n^*)$ is an optimal solution of the following LP problem:*

$$\begin{aligned} &\textit{maximize} \quad c_1 x_1 + \cdots + c_n x_n \\ &\textit{subject to} \quad a_{i1} x_1 + \cdots + a_{in} x_n \leq b_i, \quad i \in \mathcal{M}, \\ &\qquad\qquad\quad x_j \geq 0, \quad j \in \mathcal{N}. \end{aligned} \tag{9.66}$$

PROOF. The proof immediately follows from the definition (9.64) of the extended operations and from (6.37). ∎

By A_x we denote the set of indices of all active constraints of (9.66) at $x = (x_1, \cdots, x_n)$, i.e.,

$$A_x = \{i \in \mathcal{M} \mid a_{i1} x_1 + \cdots + a_{in} x_n = b_i\}.$$

The following proposition gives a necessary condition for the existence of a feasible solution of (9.65). The proof can be found in [60].

PROPOSITION 9.17 *Let $\mathcal{B}$ be a linearly ordered basis of generators. Let $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \in \mathcal{F}_{\mathcal{B}}(\mathbf{R})^n$ be a feasible solution of (9.65), where $\tilde{x}_j = (x_j, \xi_j)$, $j \in \mathcal{N}$. Then the vector $x = (x_1, \cdots, x_n)$ is the feasible solution of the LP problem (9.66) and it holds*

(i) *if* $\diamond = \max$ *and* $\star = \min$, *then*

$$\min\{a_{ij} \mid j \in \mathcal{N}\} \leq_{\mathcal{B}} b_i \quad \text{for all } i \in A_x;$$

(ii) *if* $\diamond = \max$ *and* $\star = \max$, *then*

$$\max\{a_{ij} \mid j \in \mathcal{N}\} \leq_{\mathcal{B}} b_i \quad \text{for all } i \in A_x.$$

In this subsection we have presented an alternative approach to LP problems with fuzzy parameters. Comparing to the approach presented in the previous sections, the decision variables x_j considered here have not been taken as crisp numbers, they have been considered as fuzzy intervals of the same type as the corresponding coefficients - parameters of the optimization problem. From the computational point of view this approach is simple as it requires to solve only a classical LP problem.

8. Illustrative Examples

In this section we present two "one-dimensional examples" illustrating the basic concepts. The examples could be, however, extended from $\mathbf{R}$ to $\mathbf{R}^n$.

EXAMPLE 9.18 Consider the following simple FLP problem in $\mathbf{R}$.

$$\begin{aligned} \text{maximize} \quad & \tilde{c}x \\ \text{subject to} \quad & \tilde{a}x \tilde{\leq} \tilde{b}, \\ & x \geq 0. \end{aligned} \tag{9.67}$$

Here, $\tilde{c}, \tilde{a}$ and $\tilde{b}$ are supposed to be crisp subsets of $\mathbf{R}$, particularly, closed bounded intervals: $\tilde{c} = [\underline{c}, \overline{c}]$, $\tilde{a} = [\underline{a}, \overline{a}]$, $\tilde{b} = [\underline{b}, \overline{b}]$, with $\underline{c}, \underline{b} > 0$. Let $T = \min$. Remember that the membership functions of $\tilde{c}, \tilde{a}$ and $\tilde{b}$ are their characteristic functions. The fuzzy relation $\tilde{\leq}$ and $\tilde{\geq}$ is assumed to be a T-fuzzy extension of the binary relation $\leq$ and $\geq$, respectively.

(I) Membership functions of $\tilde{c}x$ and $\tilde{a}x$.

By (6.15) we obtain for every $t \in \mathbf{R}$:

$$\mu_{\tilde{c}x}(t) = \sup\{\chi_{[\underline{c}, \overline{c}]}(c) \mid c \in \mathbf{R},\ cx = t\} = \begin{cases} 1 & \text{if } \underline{c}x \leq t \leq \overline{c}x, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we obtain the membership function of $\tilde{a}x$ as

$$\mu_{\tilde{a}x}(t) = \sup\{\chi_{[\underline{a},\bar{a}]}(a) \mid a \in \mathbf{R},\ ax = t\} = \begin{cases} 1 & \text{if } \underline{a}x \leq t \leq \bar{a}x, \\ 0 & \text{otherwise.} \end{cases} \tag{9.68}$$

Now, we derive the membership function $\mu_{\tilde{\leq}}$, $\mu_{\tilde{\geq}}$ of the fuzzy relations $\tilde{\leq}$, $\tilde{\geq}$, respectively.

By (6.29) we obtain

$$\begin{aligned} \mu_{\tilde{\geq}}(\tilde{c}y, \tilde{c}x) &= \sup\{\min\{\mu_{\tilde{c}y}(u), \mu_{\tilde{c}x}(v)\} \mid u \geq v\}, \\ \mu_{\tilde{\leq}}(\tilde{a}x, \tilde{b}) &= \sup\{\min\{\mu_{\tilde{a}x}(u), \mu_{\tilde{b}}(v)\} \mid u \leq v\}. \end{aligned} \tag{9.69}$$

A feasible solution can be calculated as follows.

By (9.7) a feasible solution $\tilde{X}$ of the FLP problem (9.67) is given by the membership function

$$\mu_{\tilde{X}}(x) = \min\{\mu_{\tilde{\leq}}(\tilde{a}x, \tilde{b}), \chi_{[0,+\infty)}(x)\} \tag{9.70}$$

By (9.69) and (9.68), we get

$$\begin{aligned} \mu_{\tilde{\leq}}(\tilde{a}x, \tilde{b}) &= \sup\{\min\{\chi_{[\underline{a}x,\bar{a}x]}(u), \chi_{[\underline{b},\bar{b}]}(v)\} \mid u \leq v\} \\ &= \begin{cases} 1 & \text{if } \underline{a}x \leq \bar{b}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{9.71}$$

Consider three cases of the value of $\underline{a}$:

Case 1: $\underline{a} > 0$. From (9.71) it follows that

$$\mu_{\tilde{\leq}}(\tilde{a}x, \tilde{b}) = \chi_{[0,\bar{b}/\underline{a}]}(x). \tag{9.72}$$

By (9.70) and (9.72) we get

$$\mu_{\tilde{X}}(x) = \min\{\chi_{[0,\bar{b}/\underline{a}]}(x), \chi_{[0,+\infty)}(x)\} = \chi_{[0,\bar{b}/\underline{a}]}(x), \tag{9.73}$$

or, in other words,

$$\tilde{X} = [0, \bar{b}/\underline{a}].$$

Case 2: $\underline{a} = 0$. Since $\bar{b} > 0$, apparently by (9.70) and (9.71) we get

$$\mu_{\tilde{X}}(x) = \chi_{[0,+\infty)}(x), \tag{9.74}$$

or

$$\tilde{X} = [0, +\infty).$$

Case 3: $\underline{a} < 0$, then $\bar{b}/\underline{a} < 0$. From (9.75) and (9.71) it follows that

$$\mu_{\tilde{\leq}}(\tilde{a}x, \tilde{b}) = \chi_{[\bar{b}/\underline{a},+\infty)}(x). \tag{9.75}$$

By (9.70) and (9.75) we get for all $x \geq 0$

$$\mu_{\tilde{X}}(x) = \chi_{[0,+\infty)}(x),$$

or

$$\tilde{X} = [0, +\infty).$$

(II) Optimal solution $\tilde{X}^*$ of FLP problem (9.67).

Consider a fuzzy goal $\tilde{d}$ given by the membership function

$$\mu_{\tilde{d}}(t) = \max\{0, \min\{\beta t, 1\}\},$$

where β is sufficiently small positive number, e.g., $\beta \leq \underline{a}/\overline{b}$, to secure that $\mu_{\tilde{d}}$ is strictly increasing function in a sufficiently large interval. By (9.20) and (9.21) we obtain

$$\mu_{\tilde{X}^*}(x) = \min\{\mu_{\tilde{X}_0}(x), \mu_{\tilde{X}}(x)\}, \tag{9.76}$$

$$\mu_{\tilde{X}_0}(x) = \mu_{\tilde{\geq}}(\tilde{a}x, \tilde{b}) = \sup\{\min\{\chi_{[\underline{c}x,\overline{c}x]}(u), \mu_{\tilde{d}}(v)\} \mid u \geq v\} = \mu_{\tilde{d}}(\overline{c}x). \tag{9.77}$$

Consider 2 cases corresponding to the value of $\underline{a}$:

Case 1: *Case 1:* $\underline{a} > 0$. Then by (9.73), $\tilde{X} = [0, \overline{b}/\underline{a}]$ and by (9.76) and (9.77) we obtain

$$\mu^*_{\tilde{X}}(x) = \begin{cases} \mu_{\tilde{d}}(\overline{c}x) & \text{if } x \in [0, \overline{b}/\underline{a}], \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha \in (0, 1]$. By Proposition 9.6, it is easy to verify that

$$[\tilde{X}^*]_\alpha = [\alpha/\beta, \overline{b}/\underline{a}].$$

By Proposition 9.7 we obtain the unique optimal solution with maximal height

$$x^* = \overline{b}/\underline{a}.$$

Case 2: $\underline{a} \leq 0$. Then by (9.74), $\tilde{X} = [0, +\infty)$ and

$$\mu^*_{\tilde{X}}(x) = \mu_{\tilde{b}_0}(\overline{c}x).$$

for all $x \in \mathbf{R}$.

Again, by Proposition 9.6 we obtain the α-cut of the optimal solution

$$[\tilde{X}^*]_\alpha = [\alpha/\overline{c}\beta, +\infty).$$

The set of all optimal solution with maximal height is the interval

$$[1/\bar{c}\beta, +\infty)\,.$$

□

EXAMPLE 9.19 Consider the same FLP problem as in Example 9.18, but with different fuzzy parameters. The problem is as follows

$$\begin{array}{ll} \text{maximize} & \tilde{c}x \\ \text{subject to} & \tilde{a}x \tilde{\leq} \tilde{b}, \\ & x \geq 0. \end{array} \tag{9.78}$$

Here, the parameters $\tilde{c}$, $\tilde{a}$ and $\tilde{b}$ are supposed to be triangular fuzzy numbers. To reduce the number of particular cases, we suppose that

$$0 < \gamma < c,\ 0 < \alpha < a,\ 0 < \beta < b.$$

Piecewise linear membership functions $\mu_{\tilde{c}}$, $\mu_{\tilde{a}}$ and $\mu_{\tilde{b}}$ are defined for each $x \in \mathbf{R}$ as follows:

$$\mu_{\tilde{c}}(x) = \max\left\{0, \min\left\{1 - \frac{c-x}{\gamma}, 1 + \frac{c-x}{\gamma}\right\}\right\}, \tag{9.79}$$

$$\mu_{\tilde{a}}(x) = \max\left\{0, \min\left\{1 - \frac{a-x}{\alpha}, 1 + \frac{a-x}{\alpha}\right\}\right\}, \tag{9.80}$$

$$\mu_{\tilde{b}}(x) = \max\left\{0, \min\left\{1 - \frac{b-x}{\beta}, 1 + \frac{b-x}{\beta}\right\}\right\}, \tag{9.81}$$

see Figure 9.1. Let $T = \min$. The fuzzy relation $\tilde{\leq}$ is assumed to be a T-fuzzy

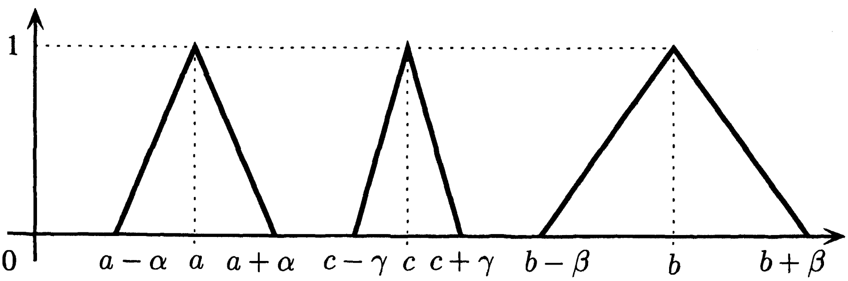

Figure 9.1.

extension of the binary relation $\leq$.

(I) Membership functions of $\tilde{c}x$ and $\tilde{a}x$.

Let $x > 0$. Then by (6.15) we obtain for every $t \in \mathbf{R}$:

$$\begin{aligned} \mu_{\tilde{c}x}(t) &= \sup\{\mu_{\tilde{c}}(c) \mid c \in \mathbf{R}, cx = t\} \\ &= \mu_{\tilde{c}}(t/x) \\ &= \max\left\{0, \min\left\{1 - \frac{cx - t}{\gamma x}, 1 + \frac{cx - t}{\gamma x}\right\}\right\}. \end{aligned} \tag{9.82}$$

In the same way, we obtain the membership function of $\tilde{a}x$ as

$$\mu_{\tilde{a}x}(t) = \max\left\{0, \min\left\{1 - \frac{ax - t}{\alpha x}, 1 + \frac{ax - t}{\alpha x}\right\}\right\} \tag{9.83}$$

Let $x = 0$. Then

$$\mu_{\tilde{c}x}(t) = \chi_0(t) \text{ and } \mu_{\tilde{a}x}(t) = \chi_0(t)$$

for every $t \in \mathbf{R}$.

Second, we calculate the membership function $\mu_{\tilde{\leq}}$ of the fuzzy relation $\tilde{\leq}$. Let $x > 0$. Then, see Figure 9.2,

$$\mu_{\tilde{\leq}}(\tilde{a}x, \tilde{b}) = \sup\{\min\{\mu_{\tilde{a}x}(u), \mu_{\tilde{b}}(v)\} \mid u \leq v\},$$

For $x = 0$ we calculate

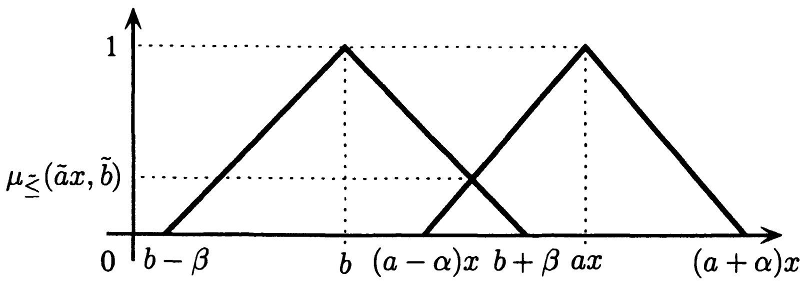

Figure 9.2.

$$\mu_{\tilde{\leq}}(\tilde{a}x, \tilde{b}) = \sup\{\min\{\chi_{\{0\}}(u), \mu_{\tilde{b}}(v)\} \mid u \leq v\} = 1.$$

(II) Feasible solution.

By (9.7) a feasible solution $\tilde{X}$ of the FLP problem (9.78) is given by the membership function

$$\mu_{\tilde{X}}(x) = \min\{\mu_{\tilde{\leq}}(\tilde{a}x, \tilde{b}), \chi_{[0,+\infty)}(x)\}. \tag{9.84}$$

Suppose that $x \geq 0$. Using (9.79), (9.80) and (9.83), we calculate

$$\begin{aligned}\mu_{\lesssim}(\tilde{a}x, \tilde{b}) &= \sup\{\min\{\mu_{\tilde{a}x}(u), \mu_{\tilde{b}}(v)\} \mid u \leq v\} \\ &= \begin{cases} 1 & \text{if } 0 < x,\, ax \leq b, \\ \frac{b+\beta-(a-\alpha)x}{\alpha x+\beta} & \text{if } b < ax,\, (a-\alpha)x \leq b+\beta, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

By (9.7) a feasible solution $\tilde{X}$ of the FLP problem (9.78) is given by the membership function

$$\mu_{\tilde{X}}(x) = \min\{\mu_{\lesssim}(\tilde{a}x, \tilde{b}), \chi_{[0,+\infty)}(x)\}.$$

Recalling (9.84), (9.87), we summarize, see Figure 9.3.

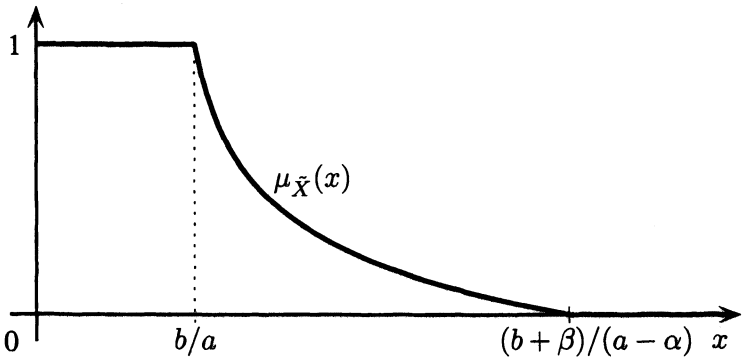

Figure 9.3.

$$\mu_{\tilde{X}}(x) = \begin{cases} 1 & \text{if } 0 \leq x,\, ax \leq b, \\ \frac{b+\beta-(a-\alpha)x}{\alpha x+\beta} & \text{if } b < ax,\, (a-\alpha)x \leq b+\beta, \\ 0 & \text{otherwise.} \end{cases} \tag{9.85}$$

Let $\varepsilon \in (0,1]$. From (9.85) it follows that

$$\mu_{\tilde{X}}(x) \geq \varepsilon$$

if and only if

$$\frac{b+\beta-(a-\alpha)x}{\alpha x+\beta} \geq \varepsilon \text{ and } x \geq 0,$$

or equivalently,

$$0 \leq x \leq \frac{b+(1-\varepsilon)\beta}{a-(1-\varepsilon)\alpha}.$$

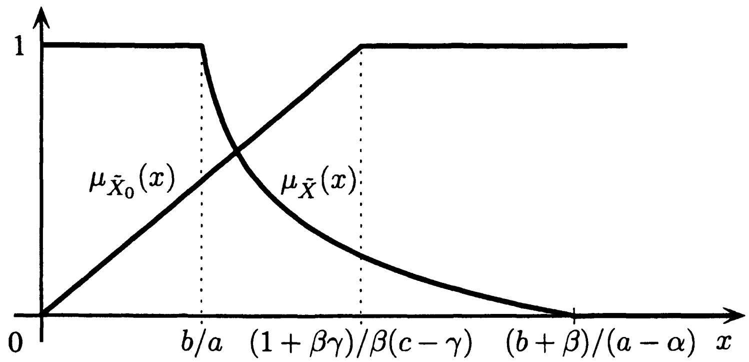

Figure 9.4.

In other words,

$$[\tilde{X}]_\varepsilon = \left[0, \frac{b+(1-\varepsilon)\beta}{a-(1-\varepsilon)\alpha}\right]. \tag{9.86}$$

(III) Optimal solution.

Consider a fuzzy goal $\tilde{d}$ given by the membership function

$$\mu_{\tilde{d}}(x) = \min\{\beta x, 1\}, \tag{9.87}$$

for all $x \geq 0$, where β is sufficiently small positive number, e.g., $\beta \leq a/b$. By (9.20) and (9.21) we have

$$\mu_{\tilde{X}^*}(x) = \min\{\mu_{\tilde{X}_0}(x), \mu_{\tilde{X}}(x)\}, \tag{9.88}$$

$$\mu_{\tilde{X}_0}(x) = \mu_{\tilde{\geq}}(\tilde{c}x, \tilde{d}) = \sup\{\min\{\mu_{\tilde{c}x}(u), \mu_{\tilde{d}}(v)\} \mid u \geq v\}. \tag{9.89}$$

By (9.82) and (9.87) we calculate for all $x \geq 0$

$$\mu_{\tilde{X}_0}(x) = \min\{\delta x, 1\}, \tag{9.90}$$

where

$$\delta = \frac{\beta(c+\gamma)}{1+\beta\gamma}.$$

The membership function of optimal solution given by (9.88) is depicted in Figure 9.4. Combining (9.86) and (9.90) we obtain the set of all max-optimal

solution $\bar{X}$ from formula (9.88) as

$$\bar{X} = \begin{cases} \frac{\sqrt{D}-[\beta\delta+(a-\alpha)]}{2\alpha\delta} & \text{if } a/b < 1/\delta, \\ [1/\delta, a/b] & \text{otherwise.} \end{cases}$$

where

$$D = [\beta\delta + (a-\alpha)]^2 + 4\alpha\delta(b+\beta),$$

see Figure 9.4. □

Chapter 10

FUZZY SEQUENCING AND SCHEDULING

1. Introduction

The early theoretical models of practical machine scheduling and sequencing problems and their mathematical analysis were motivated mostly by problems arising in manufacturing and service industries. Later models have also been influenced by new developments in flexible manufacturing, computer systems and communication systems. Analogous problems arise, however, almost everywhere, and the number of various problem types is practically unlimited. Nevertheless, the resulting mathematical and computational problems often have the form of the following optimization problem: Given a finite number of jobs to be processed on a finite number of machines, find a feasible schedule that minimizes the value of a given objective function.

To establish a clear framework for further analysis and discussion, we first recall elementary concepts and basic models of deterministic *machine scheduling*. Then we discuss some of them in nondeterministic situations with the emphasis on aspects that may be of special interest to the fuzzy set community. Starting with stochastic models, we present motivation examples characterizing difficulties which may occur under uncertainty of some parameters of the problem. Then we turn our attention to fuzzy models studying fuzzy due dates, fuzzy processing times and fuzzy precedence relations. Finally we discuss some directions of future research in fuzzy machine scheduling.

We are not concerned with the *project scheduling problems*. The reader interested in the project scheduling problems in which the uncertainty in activity durations is modeled by fuzzy quantities should consult Chapter 9 in [118].

2. Deterministic Models

We begin with the simplest machine environment, namely, with the *single machine problems*. We assume that there are n jobs $J_1, J_2, \ldots, J_n$ to be scheduled for processing by a given machine within a given scheduling period under the following two assumptions:

- The machine is always available. In other words, the machine never breaks down and is available for job processing throughout a given scheduling period. For simplicity we assume that the scheduling period is the interval of all nonnegative real numbers.
- The machine cannot process more than one job at a time. Thus, at each point in time, the machine is either idle or is processing one of the available jobs.

These assumptions make it possible to represent schedules with piecewise constant functions S that map the scheduling period into the set $\{0, 1, \ldots, n\}$ by agreeing on the following interpretation of values $S(t)$: For each t from the scheduling period,

$$S(t) = \begin{cases} k & \text{if job } J_k \text{ is processed at time } t, \\ 0 & \text{if no job is processed at time } t. \end{cases}$$

By identifying schedules with piecewise constant functions, we obtain a well defined concept, which conveys all necessary information and can easily be graphically represented.

An important observation is that each schedule S determines uniquely starting time $B_k(S)$ and completion time $C_k(S)$ of job J_k in schedule S by

$$\begin{aligned} B_k(S) &= \inf\{t \mid S(t) = k\}, \\ C_k(S) &= \sup\{t \mid S(t) = k\}. \end{aligned}$$

Another important observation is that not every piecewise constant function S that map the scheduling period into the set $\{0, 1, \ldots, n\}$ represents a feasible schedule because a number of additional, problem dependent, requirements may restrict the feasibility of schedules.

For example, quite often a job is required to be fully processed before some other jobs may be begun (*precedence constraints*). However, the most basic additional assumption arises from the requirement that every job must spend on the machine a prescribed length of time. In other words, we assume that there are given positive numbers $p_1, p_2, \ldots, p_n$, so called processing times, and every job J_k must spend on the machine p_k units of time to be completed. This assumption does not specify whether the processing of a job can be interrupted and resumed later. In this respect two types of problems are usually

distinguished: those in which the jobs must be processed *nonpreemptively* and those in which job *preemption* is permitted.

Other types of restriction arise in situations where a job becomes available for processing later than at the beginning of the scheduling period or a job must be completed earlier than by the end of the scheduling period. The presence of such job *release times* and job *deadlines* makes scheduling problems so difficult computationally that already the problem of deciding whether a feasible schedule exists is NP-hard if the jobs must be scheduled nonpreemptively. The reader who is not familiar with the concept of NP-hardness and other basic concepts of the computational complexity can consult the book [35]. In what follows we deal only with the nonpreemptive scheduling under the assumptions that there are no deadlines and all jobs are available for processing form the beginning of the scheduling period. As a consequence, the question of the existence of a feasible schedule is trivial - there is infinite number of feasible schedules.

Since there are many feasible schedules, the problem becomes that of finding a *feasible schedule* which is best in some prescribed sense. To specify the meaning of *best*, some suitable tools are needed for the mutual comparison of schedules. The standard approach is to compare schedules by means of a real-valued function f defined on the set of feasible schedules and constructed in such a way that $f(S) < f(S')$ whenever schedule S is considered to be better than schedule S'. Then the problem of finding the best schedule among the set of feasible schedules becomes the problem of minimizing $f(S)$ over the set of feasible schedules.

Such *objective functions* are often defined with the help of extremely simple quantities involving *due dates*. In contrast to the notion of a deadline, a job can be processed after its due date but this may incur additional cost or penalty. This additional cost is usually expressed as a function of the difference between the completion time and due date of the job. This difference is called *lateness* in spite of the possibility that the job is early and not late when it is completed before its due date. Such negative lateness may represent better service than required. When early jobs bring no reward, then the *tardiness* defined as the positive part of the lateness is a more appropriate quantity for constructing the objective function. In some situations, for instance in a just-in-time environment, also negative lateness may represent poorer service than required, and it should be penalized.

Typically, objective functions are composed from these simple quantities through aggregation by summation or maximization. More generally, let $\mathcal{N} = \{1, 2, \ldots, n\}$, and assume that a real-valued function φ_k is associated with job J_k, $k \in \mathcal{N}$. Then these given penalty or *cost functions* are aggregated into the

objective function f_{sum} or $f_{\max}$ as follows:

$$f_{\text{sum}}(S) = \sum_{k\in\mathcal{N}} \varphi_k(C_k(S)), \tag{10.1}$$

$$f_{\max}(S) = \max\left\{\varphi_k(C_k(S)) \mid k \in \mathcal{N}\right\}. \tag{10.2}$$

For example, if we use (10.1) with cost functions

$$\varphi_k(t) = \frac{1}{n}(t - d_k), \quad k \in \mathcal{N},$$

where d_k is the due date of J_k, then we obtain the problem of minimizing the *mean lateness*; if we use (10.1) with cost functions

$$\varphi_k(t) = \max\{0, \text{sign}(t - d_k)\}, \quad k \in \mathcal{N},$$

then we have the problem of minimizing the *number of tardy jobs*.

A very useful property of objective functions defined by (10.1) and (10.2) is that if all cost functions φ_k are nondecreasing, then both f_{sum} and $f_{\max}$ are monotone in the sense that $f(S) \leq f(S')$ whenever $C_k(S) \leq C_k(S')$ for all $k \in \mathcal{N}$. The objective functions that have this property are called *regular objective functions* or *regular measures of performance*. Note that the regularity can be guaranteed for much broader class of objective functions by using the monotonicity of aggregating mappings discussed in Chapter 5.

The significance of this type of regularity consists in the fact that, in certain situations, the search for an optimal schedule can be limited to a small part of the set of all feasible schedules. For example, under our assumption that all jobs are available from the beginning of the scheduling period, no improvement in the optimal value of a regular objective function can be gained by allowing inserted *idle times* between the processing of jobs or by preemption. In other words, it is sufficient to minimize only over the set of those schedules in which the machine starts processing at the beginning of the scheduling period and continues without interruption and preemption until all jobs are completed. Such schedules are called *permutation schedules* because each such schedule is uniquely determined by the order (permutation) in which the jobs are processed.

For later reference, we recall two well known algorithms that take advantage of objective function regularity.

LAWLER'S ALGORITHM [65]

STEP 1: Given n jobs with positive processing times $p_1, p_2, \ldots, p_n$, set

$$I := \mathcal{N}, \; u := \sum_{j\in I} p_j, \; k := n.$$

STEP 2: For given cost functions $\varphi_1, \varphi_2, \ldots, \varphi_n$, select $i \in I$ such that

$$\varphi_i(u) = \min \{\varphi_j(u) \mid j \in I\},$$

and place J_i in the kth position.

STEP 3: Update I, u and k by

$$I := I \setminus \{i\}, \quad u := u - p_i, \quad k := k - 1,$$

and return to Step 2 until $I = \emptyset$.

It can be verified that Lawler's algorithm delivers an optimal permutation for the problem with the objective function $f_{\max}$ defined by (10.2) whenever all cost functions φ_k are nondecreasing. In spite of its generality, Lawler's algorithm is quite efficient; it can be run in $O(n^2)$-time, provided that the values of the cost functions can be computed in constant time. For simplicity, we presented the algorithm for the case of *independent jobs*, that is, for the case of empty precedence relation. However, the algorithm can easily be extended to the case where the feasibility of schedules is constrained by a precedence relation given by an irreflexive partial ordering on the job set.

The following simple algorithm solves the problem of minimizing the number of tardy jobs in $O(n \log n)$-time where n denotes the number of jobs.

MOORE'S ALGORITHM [73]

STEP 1: Given n jobs with positive processing times $p_1, p_2, \ldots, p_n$ and nonnegative due dates $d_1, d_2, \ldots, d_n$, sequence the jobs according to nondecreasing due dates.

STEP 2: Find the first tardy job in the current sequence, say J_k. If no such job exists, go to Step 4.

STEP 3: Find the longest job in the initial part of the sequence up to and including J_k, reject it from the sequence, and return to Step 2.

STEP 4: Form the final sequence by taking the current sequence and appending to it the rejected jobs in arbitrary order.

We should note that an objective function, regular or not, is a tool for representing an underlying relation *better than*, and that some natural relations cannot be represented by a single objective function. As typical examples of such situations we can consider the problems involving comparison of schedules with respect to several conflicting objectives. Several functions can be used to deal with such situations, which results in a scheduling problem with

multiple objectives. The standard way is to construct a finite number of real-valued functions $f_1, f_2, \ldots f_p$ such that the inequalities

$$\begin{aligned} f_i(S) &\le f_i(S') \quad \text{for all } 1 \le i \le p, \\ f_i(S) &< f_i(S') \quad \text{for at least one } 1 \le i \le p \end{aligned}$$

are satisfied if and only if schedule S is considered to be better than schedule S'. The problem then becomes that of finding the set of Pareto-minimizers with respect to functions $f_1, f_2, \ldots, f_p$ over the set of feasible schedules.

One of the immediate extensions of single-machine models takes advantage of parallelism. We again assume that there are n jobs $J_1, J_2, \ldots, J_n$ to be processed, but now on a system of m *machines* $M_1, M_2, \ldots, M_m$. All machines are again always available throughout a given scheduling period, and this period is machine independent. Each machine can process each job but again not more than one job at a time. Moreover no job can be processed simultaneously on different machines. Since the machines are not necessarily identical, the processing time of a job may vary from machine to machine. Let $\mathcal{M} = \{1, 2, \ldots, m\}$, and let $p_{ik} > 0$, $i \in \mathcal{M}$, $k \in \mathcal{N}$, denote the processing time of job J_k if it is processed only by machine M_i. Under these assumptions, each feasible schedule can be represented by an m-tuple $S = (S_1, S_2, \ldots, S_m)$ of schedules for individual machines. However not every such m-tuple represents a feasible schedule. If $L_{ik}(S)$ denotes the "length" of the set $\{t \mid S_i(t) = k\}$, then the requirement of processing all jobs to completion can be expressed by the condition

$$\sum_{i \in \mathcal{M}} \frac{L_{ik}(S)}{p_{ik}} = 1, \quad k \in \mathcal{N}.$$

Moreover, since each job can be processed at most by one machine at a time, we have to require that $S_i(t) = k$ implies $S_j(t) \ne k$ for each job J_k and for every j different from i.

Usually we distinguish the following three situations. If there are positive numbers $p_1, p_2, \ldots, p_n$ such that $p_{1k} = p_{2k} = \cdots p_{mk} = p_k$ for each $k \in \mathcal{N}$, then we say that our parallel machines are *identical*. If there are positive numbers $p_1, p_2, \ldots, p_n$ and $s_1, s_2, \ldots, s_n$ such that $p_{ik} = s_i p_k$ for $i \in \mathcal{M}$ and $k \in \mathcal{N}$, then we speak about *uniform* parallel machines. If there is no special relation between various processing times, we say that the machines are *unrelated*.

As in the single-machine case, each feasible schedule determines, for each job J_k, completion time $C_k(S)$ of job J_k under the schedule S. Consequently, everything concerning objective functions based on completion times (for example the concept of regular objective functions) can easily be extended to parallel-machine models.

Single-machine and *parallel-machine models* can be considered as single operation or single stage problems in the sense that each job consists of just one operation and has no inner structure. There are of course many problems of practical interest in which jobs are more complex and their processing requires more intricate system of machines.

It is an established tradition to distinguish three basic types of such multi-operation models: *flow shops*, *job shops*, and *open shops*. In all three cases we again have n jobs $J_1, J_2, \ldots, J_n$ to be processed on the system of m machines $M_1, M_2, \ldots, M_m$. However, now each job J_k consists of m operations $O_{1k}, O_{2k}, \ldots, O_{mk}$. In an *open shop*, each operation O_{ik} has to be processed by machine M_i but no particular processing order through the machines is prescribed. In a *flow shop*, a processing order through the machines is given and this order is the same for all jobs. In a *job shop*, the order through the machines is also prescribed for each job, but it may be different for different jobs. Again we suppose that each machine can handle at most one job at a time and no job can be processed by more than one machine simultaneously. Each operation O_{ik} has its own positive processing time p_{ik} depending on the machine which must execute the operation.

Again, each feasible schedule determines uniquely the completion time of jobs. Therefore the concept of a regular objective function and all other concepts based on completion times can easily be extended to multi-operation models. Among the traditional multi-operation problems, the job shop problems seem to be hardest. A theoretical proof of their intractability comes from the fact that even the problem of minimizing $f_{\max}$ with $\varphi_k(t) = t$ for all $k \in \mathcal{N}$ is NP-hard in the strong sense. A practical proof comes from the fact that it took more than 10 years to solve a particular "small" instance of this problem with 10 jobs and 10 machines posed in 1963, see [30].

There are several excellent books and surveys on deterministic machine scheduling. We refer the reader to the books [4], [16], [23], [34], [80]. For survey of complexity results, optimization algorithms and approximation algorithm, see [66].

3. Stochastic Models

In the previous part of this chapter, we assume that the processing times, due dates and all other data defining a problem instance are known exactly in advance. In practice, we are often confronted with situations where this assumption is not justified because various forms of uncertainty must be taken into account. For example, uncertainty can arise because we are unable to measure some quantities perfectly or because some operations involve human activities or depend on weather. In this section we illustrate how situations involving uncertainty are treated if a probabilistic description of uncertain parameters is available.

Mathematical models based on probability theory have been used to quantify uncertain data to facilitate formulation, analysis and solution of machine scheduling problems since the beginning of scheduling theory. Some of such stochastic problems have been studied in the framework of queueing theory or stochastic dynamic programming. However, gradually, the field of stochastic machine scheduling has been established with its specific methods, applications and vast literature. The reader interested in this field can consult, for instance, the book [81] or the survey [107]. Here, similarly to the previous sections, we aim at providing a motivation for the next part devoted to the models based on fuzzy set theory.

We begin with considering some stochastic counterparts of single machine problems discussed in the previous section. We assume that all problem data except the processing times are known with certainty in advance. Regarding the machine, we exclude the possibility of machine breakdowns, and again assume that the machine can process at most one job at a time. The only uncertainty arises from the assumption that the processing times are independent random variables whose distributions are known in advance.

Randomness in the processing times implies distributions of job completion times, which makes the meaning of "better than" relation unclear. Obviously, some kind of stochastic dominance should be specified to make mutual comparison of schedules possible. The simplest and quite common approach is to replace the values of objective functions used in the deterministic case with their expected values. Let us illustrate it on a simple example. The reader interested in applications of other types of stochastic dominance finds plenty of examples in [107], [81] and [19].

EXAMPLE 10.1 Consider two jobs J_1 and J_2 whose due dates are $d_1 = 2$ and $d_2 = 5$, respectively, and whose processing times are discrete random variables with probability distributions given by the following tables:

Processing time of J_1	*Probability*
1	1/3
2	1/3
3	1/3

Processing time of J_2	*Probability*
2	1/2
4	1/2

The problem is to establish which of the two possible permutation schedules is better when the measure of performance to be minimized is the expected value of the maximum lateness.

Let L_1 and L_2 denote the lateness of J_1 and J_2, respectively. There are two cases: the case that job J_1 is scheduled first and the case that job J_2 is scheduled first. Simple calculations show that the probability distributions of $\max\{L_1, L_2\}$ in these two cases are respectively as follows:

Case: job J_1 is scheduled first.

Value of $\max\{L_1, L_2\}$	*Probability*
-1	$1/6$
0	$2/6$
1	$2/6$
2	$1/6$

Case: job J_2 is scheduled first.

Value of $\max\{L_1, L_2\}$	*Probability*
1	$1/6$
2	$1/6$
3	$2/6$
4	$1/6$
5	$1/6$

It follows that the expected value of the maximum lateness is

$$E(\max\{L_1, L_2\}) = \begin{cases} 1/2 & \text{if } J_1 \text{ is scheduled first,} \\ 3 & \text{if } J_2 \text{ is scheduled first.} \end{cases}$$

Thus we conclude that the schedule in which job J_1 is scheduled first is better.

□

The example suggests that if, under the same assumptions, a finite set of jobs is to be scheduled with the objective to minimize the expected value of the maximum lateness, then the best schedule could be obtained by ordering the jobs in nondecreasing order of job due dates (*EDD-rule*). This is indeed so, see [81].

EXAMPLE 10.2 Consider the same job data as in the previous example and calculate the expected values of the lateness of jobs J_1 and J_2 in the case that job J_1 is scheduled first. Again easy calculation gives the following probability distributions of L_1 and L_2:

Value of L_1	*Probability*
-1	$1/3$
0	$1/3$
1	$1/3$

Value of L_2	*Probability*
-2	$1/6$
-1	$1/6$
0	$2/6$
1	$1/6$
2	$1/6$

Obviously the expected values of L_1 and L_2 are $E(L_1) = E(L_2) = 0$. Observe that

$$\max\{E(L_1), E(L_2)\} < E(\max\{L_1, L_2\}).$$

□

This example indicates that the stochastic counterparts of the minimization of the maximum lateness in a deterministic environment may differ according to whether we wish to minimize the expected value of the maximum lateness or the maximum of the expected job latenesses. It turns out that the latter problem is also solved by the EDD-rule. However, this type of problems can

be solved by a modification of Lawler's algorithm for arbitrary nondecreasing cost functions. It suffices to modify the second step of the algorithm as follows:

STEP 2*: Select $i \in I$ such that for given cost functions $\varphi_1, \varphi_2, \ldots, \varphi_n$,

$$\int_0^\infty \varphi_i(t) f_I(t) dt = \min \left\{ \int_0^\infty \varphi_j(t) f_I(t) dt \;\middle|\; j \in I \right\}$$

where f_I denotes the convolution of the processing times of jobs with indices in I.

Obviously, the efficiency of the algorithm depends on the ability to evaluate the integrals. The properties of expected values imply that this task is particularly easy if the cost functions are linear.

The previous two examples illustrate the difference between the minimization of the function

$$f_{E,\max}(S) = E(\max\{\varphi_k(C_k(S)) \mid k \in \mathcal{N}\})$$

and the minimization of the function

$$f_{\max,E}(S) = \max\{E(\varphi_k(C_k(S))) \mid k \in \mathcal{N}\}$$

over the set of permutation schedules. Both these variants are stochastic counterparts of the deterministic problem whose objective function is based on the aggregation of cost functions through maximization.

If the aggregation of cost functions is based on summation, then the difference between the minimization of

$$f_{E,\Sigma}(S) = E\left(\sum_{k \in \mathcal{N}} \varphi_k(C_k(S))\right)$$

and the minimization of

$$f_{\Sigma,E}(S) = \sum_{k \in \mathcal{N}} \varphi_k(E(C_k(S)))$$

can be much more drastic. As an convincing example, consider the problem of scheduling nonpreemptively n jobs $J_1, J_2, \ldots, J_n$ for processing by a single machine with the objective to minimize the weighted number of tardy jobs under the assumption that all jobs have common positive due date d.

The objective function of the deterministic version is

$$f_{\text{sum}}(S) = \sum_{k \in \mathcal{N}} \varphi_k(C_k(S))$$

with cost functions

$$\varphi_k(t) = \max\{0, w_k \mathrm{sign}(t - d)\}, \quad k \in \mathcal{N},$$

where w_ks are given positive weights.

This problem is equivalent to the *knapsack problem.* Indeed, the common due date is equivalent to the size of the knapsack, the processing times are equivalent to the sizes of items, and the weights are equivalent to the values of items in the knapsack. Therefore the problem is NP-hard.

Now consider the problem of minimization of the expected weighted number of tardy jobs, that is, the minimization of the function

$$f_{E,\mathrm{sum}}(S) = E\left(\sum_{k \in \mathcal{N}} \max\{0, w_k \mathrm{sign}(C_k(S) - d)\}\right)$$

over the set of permutation schedules. In [81] it is shown that if every job J_j has an exponentially distributed processing time with rate λ_j, then the problem is solved by the following simple rule: *Sequence the jobs in nonincreasing order of* $\lambda_j w_j$.

However, if the problem is to minimize the weighted number of jobs expected to be tardy, that is, if the objective function is

$$f_{\mathrm{sum},E}(S) = \sum_{k \in \mathcal{N}} \max\{0, w_k \mathrm{sign}(E(C_k(S)) - d)\},$$

then the problem is again equivalent to the knapsack problem and therefore it is NP-hard, see [122].

An important distinction between the deterministic machine scheduling and stochastic machine scheduling consists in the possibility of adapting schedules to evolution of stochastic processes. So far we have tacitly assumed that, similarly to the deterministic environment, the order of jobs in a schedule is fixed at the beginning of the scheduling period and that this order is not changed during the processing. Since the processing times are random variables, we can consider also an alternative concept of a schedule. Namely, we may allow to change the order of unprocessed jobs at every point in time (not known in advance) at which some uncertain data became known.

Let us clarify the difference between these two (static and dynamic) concepts of a schedule through a simple example.

EXAMPLE 10.3 Consider the problem of nonpreemptively scheduling three jobs with common due date $d = 7$ and with random processing times given by the following probability distributions:

Processing time of J_1	Probability
2	4/10
3	6/10

Processing time of J_2	Probability
5	3/10
6	7/10

Processing time of J_3	Probability
2	2/10
6	8/10

First let us calculate the expected number of tardy jobs for the static schedules in which job J_1 is scheduled first. There are two such permutation schedules: One with the job order $\langle J_1, J_2, J_3 \rangle$ and the other with the job order $\langle J_1, J_3, J_2 \rangle$. For both schedules, J_1 is tardy with probability 0. For the former, J_2 is tardy with probability 22/25 (1 minus the probability that the processing time of J_1 is 2 times the probability that the processing time of J_2 is 5), and J_3 is tardy with probability 1. Thus, the expected number of tardy jobs is

$$0(0) + 1(3/25) + 2(22/25) + 3(0) = 1.88.$$

For the latter, J_3 is tardy with probability 4/5 (the probability that the processing time of J_3 is 6), and J_2 is tardy with probability 1. Thus, the expected number of tardy jobs is

$$0(0) + 1(1/5) + 2(4/5) + 3(0) = 1.8.$$

Now let us calculate the expected number of tardy jobs for the dynamic schedules in which again job J_1 is scheduled first. There are four such schedules.

1 Always schedule J_2 as the second job (i.e., the job order is always $\langle J_1, J_2, J_3 \rangle$).

2 Always schedule J_3 as the second job (i.e., the job order is always $\langle J_1, J_3, J_2 \rangle$).

3 If the processing time of J_1 is 2, then schedule J_2 as the second job (i.e., the job order is $\langle J_1, J_2, J_3 \rangle$); if the processing time of J_1 is 3, then schedule J_3 as the second job (i.e., the job order is $\langle J_1, J_3, J_2 \rangle$).

4 If the processing time of J_1 is 2, then schedule J_3 as the second job (i.e., the job order is $\langle J_1, J_3, J_2 \rangle$); if the processing time of J_1 is 3, then schedule J_2 as the second job (i.e., the job order is $\langle J_1, J_2, J_3 \rangle$).

Again, for all four schedules, J_1 is tardy with probability 0. For the first two schedules, the expected numbers of tardy jobs are 1.88 and 1.8, respectively. For the third schedule, there are exactly two tardy jobs:

- if the processing time of J_1 is 2 and the processing time of J_2 is 6, or
- if the processing time of J_1 is 3 and the processing time of J_3 is 6.

The probability that there are exactly two tardy jobs is

$$(4/10)(7/10) + (6/10)(8/10) = 19/25.$$

Thus, the expected number of tardy jobs is

$$0(0) + 1(6/25) + 2(19/25) + 3(0) = 1.76.$$

For the fourth schedule, there are exactly two tardy jobs:

- if the processing time of J_1 is 2 and the processing time of J_3 is 6, or
- if the processing time of J_1 is 3 and the processing time of J_2 is 6.

The probability that there are exactly two tardy jobs is

$$(4/10)(8/10) + (6/10)(7/10) = 37/50.$$

Thus, the expected number of tardy jobs is

$$0(0) + 1(13/50) + 2(37/50) + 3(0) = 1.74.$$

As expected, better results may be achieved by allowing for dynamic type of schedules. □

4. Fuzzy Models

For some time, machine scheduling models based on probability theory were considered the only sensible models for dealing with uncertainty in the problem data. However, the interest in scheduling models based on fuzzy set theory seems to grow rapidly, see the book [118]. We believe that more applications of fuzzy set theory to the analysis of machine scheduling are to come and that a sound theory complementing the deterministic and stochastic scheduling theories will soon emerge. Here we present only a short survey of existing approaches, discuss some new results in the area of algorithms working with fuzzy processing times and fuzzy precedences, and give some suggestions for future research.

In what follows we assume that there are n jobs $J_1, J_2, \ldots, J_n$ that have to be processed by a continuously available machine which can process at most one job at a time. Also we assume that the scheduling period is the interval of all nonnegative real numbers and all jobs are available from the beginning. In addition, we confine ourselves to nonpreemptive setting, and we assume that the only control we have is in the order in which jobs are processed. In other words, only permutation schedules are feasible. This is not always justifiable in practice. For example it may be advantageous to allow for inserted idle time if the objective function is based on just-in-time requirements.

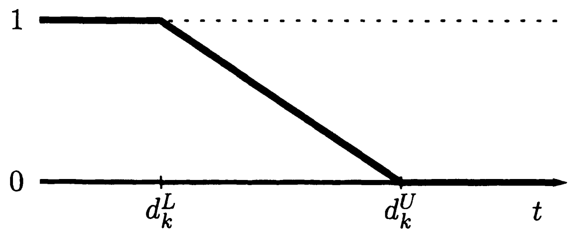

Figure 10.1.

4.1. Fuzzy Due Dates

We begin with problems in which everything remains crisp except the job due dates. We assume that the uncertainty concerning due dates is modeled by means of fuzzy intervals as defined in Chapter 6. In other words, we assume that the crisp due dates $d_1, d_2, \ldots, d_n$ of jobs $J_1, J_2, \ldots, J_n$ used to define objective functions in the deterministic case are now replaced by fuzzy intervals $D_1, D_2, \ldots, D_n$. Since the scheduling period begins at time zero, we assume that all these due dates are nonnegative in the sense that their membership functions are identically zero on the negative halfline. To simplify the notation we write $D_k(t)$ to denote the value $\mu_{D_k}(t)$ of the membership function of D_k at t.

The value $D_k(t)$ serves for expressing the degree of satisfaction with the completion of job J_k at time t, and we assume that all due dates are exactly known in advance. For example, if D_k is a right trapezoidal fuzzy interval defined by (see Figure 10.1)

$$D_k(t) = \begin{cases} 1 & \text{if } t \leq d_k^L, \\ 1 - \dfrac{t - d_k^L}{d_k^U - d_k^L} & \text{if } d_k^L < t < d_k^U, \\ 0 & \text{if } t \geq d_k^U, \end{cases} \tag{10.3}$$

where d_k^L and d_k^U are crisp real numbers such that $0 \leq d_k^L < d_k^U$, then we are completely satisfied when job J_k is completed by time $t = d_k^L$ and our degree of satisfaction decreases linearly with the job lateness to complete dissatisfaction if the job is not completed before $t = d_k^U$. The limit case with $d_k^L = d_k$ and d_k^U converging to d_k^L, that is, the case

$$D_k(t) = \begin{cases} 1 & \text{if } t \leq d_k, \\ 0 & \text{if } t > d_k, \end{cases} \tag{10.4}$$

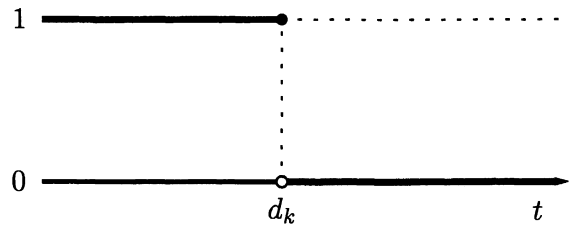

Figure 10.2.

reflects the situation in which we are completely satisfied when job J_k is completed by $t = d_k$ and completely dissatisfied when job J_k is late; see Figure 10.2.

A number of papers have appeared that deal with the problem of maximizing the minimum degree of satisfaction or the total degree of satisfaction for various classes of fuzzy due dates and machine environments. For example, in [124] the problem of maximizing the minimum degree of satisfaction is studied in the case that the fuzzy due dates are given by (10.3), that is, the problem of maximizing

$$F_{\min}(S) = \min\{D_k(C_k(S)) \mid k \in \mathcal{N}\}.$$

The algorithm presented in [124] is based on the following observations. Let $\bar{v}$ be the maximum value of $F_{\min}(S)$ over all permutation schedules, and let $\delta_k : [0, 1] \to \mathbf{R}$ be defined by

$$\delta_k(\alpha) = d_k^L + (d_k^U - d_k^L)(1 - \alpha), \quad k \in \mathcal{N}. \tag{10.5}$$

OBSERVATION 10.4 *If $\bar{v} = 0$ then, for each permutation schedule S, there exists a job J_k such that $\delta_k(\alpha) < C_k(S)$ for all $\alpha \in [0, 1]$.*

OBSERVATION 10.5 *If $\bar{v} > 0$ then $C_k(S) \leq \delta_k(\bar{v})$ for each optimal permutation schedule and each job J_k.*

OBSERVATION 10.6 *For each $\alpha \in [0, 1]$, a permutation schedule S satisfying the system of inequalities*

$$C_k(S) \leq \delta_k(\alpha), \quad k \in \mathcal{N}, \tag{10.6}$$

exists if and only if the permutation schedule in which the jobs are ordered according to nondecreasing values of $\delta(\alpha)$ satisfies (10.6).

OBSERVATION 10.7 *If I is a subinterval of the unit interval $[0,1]$ such that there are no $\alpha \in I$ and no i, j such that*

$$\alpha(d_i^U - d_i^L - d_j^U + d_j^L) = d_i^U - d_j^L,$$

then the order of $\delta_k(\alpha)$, $k \in \mathcal{N}$, does not change throughout interval I.

A simple asymptotic analysis, see [124], shows that the resulting algorithm takes $O(n^2 \log n)$-time. In [127] a modification of this algorithm is described which delivers an optimal solution in $O((P+n\log n)\log P)$-time, where P is the number of intersections of graphs of δ_ks situated between the lines $\alpha = 1$ and $\alpha = 0$.

It can easily be seen that this problem can also be solved by a straightforward application of Lawler's algorithm, because the problem is equivalent to a deterministic problem where objective function is of $f_{\max}$ type with nondecreasing cost functions. It suffices to replace every fuzzy due date D_k with its complement $\bar{D}_k$ defined by

$$\bar{D}_k(t) = 1 - D_k(t).$$

In other words, it suffices to minimize the maximum degree of dissatisfaction instead of maximizing the minimum degree of satisfaction. The problem then becomes that of minimizing $f_{\max}$ with nondecreasing cost functions

$$\varphi_k(t) = \bar{D}_k(t) = \begin{cases} 0 & \text{if } t \le d_k^L, \\ \dfrac{t - d_k^L}{d_k^U - d_k^L} & \text{if } d_k^L < t < d_k^U, \\ 1 & \text{if } t \ge d_k^U. \end{cases}$$

Obviously, analogous transformations can be used not only in the case of "right trapezoidal" due dates but also for arbitrary fuzzy intervals. Also the maximization of the minimum degree of satisfaction is not essential for such transformation. In principle, similar transformation can be used for all problems in which all data except due dates are crisp. As a consequence, all techniques of the deterministic machine scheduling are directly applicable and the study of these problems can be considered as a part of the deterministic scheduling.

4.2. Fuzzy Processing Times

The real challenge of fuzzy approach to machine scheduling begins with problems in which, in addition to fuzzy due dates, also some other problem

data are fuzzy. The reason is that in such cases fuzziness enters the constraints defining the schedule feasibility. Such situations require incorporation of deeper results of fuzzy set theory.

In this section we assume that also job processing times are allowed to be fuzzy. More precisely, we assume that each job J_k has not only a fuzzy due date D_k but also a fuzzy processing time P_k, and that both are nonnegative fuzzy intervals. For convenience, we identify feasible schedules with permutations of job indices by agreeing that if π is a permutation of the set $\mathcal{N}$, then the equality $j = \pi(i)$ indicates that job J_j is the ith job to be processed.

Similarly to the deterministic case, every schedule π determines an ordered n-tuple $(C_1^\pi, C_2^\pi, \ldots, C_n^\pi)$ of completion times. The only difference is that now every completion time is a fuzzy quantity determined for $j = \pi(i)$ by

$$C_j^\pi = P_{\pi(1)} \oplus P_{\pi(2)} \oplus \cdots \oplus P_{\pi(i)}$$

where $\oplus$ stands for a suitable addition of fuzzy intervals.

Similarly to the stochastic case, it is not clear how to compare the quality of schedules on the basis of completion times. Obviously some type of fuzzy dominance between fuzzy quantities should be used. Tens of reasonable indices for the comparison of fuzzy quantities can be found in the literature, and the right choice depends on the type of the problem under consideration.

In [53], the due dates D_k defined by (10.4) are considered, and schedules are compared by means of numbers $v_k(C_k^\pi, D_k)$ defined by

$$v_k(C_k^\pi, D_k) = \begin{cases} 1 & \text{if } B_k^\pi(D_k) \geq \lambda A_k^\pi, \\ 0 & \text{otherwise,} \end{cases}$$

where

- $\lambda \in [0, 1]$.
- A_k^π is the area below the membership function of C_k^π; see Figure 10.3.
- $B_k^\pi(D_k)$ is the area of the part of A_k^π to the right of d_k defining D_k; see Figure 10.3.

The problem considered is that of minimizing the number of λ-tardy jobs, where job J_k is called λ-tardy in sequencing according to π whenever

$$B_k^\pi(D_k) \geq \lambda A_k^\pi. \tag{10.7}$$

It is shown how Moore's algorithm could be extended for solving the problem when completion times are triangular fuzzy numbers.

In [54], the due dates D_k defined by (10.3) are considered, and schedules are compared by means of the numbers

$$v_k(C_k^\pi, D_k) := \begin{cases} 1 & \text{if } \sup_t \min\{C_k^\pi(t), D_k(t)\} \leq \lambda, \\ 0 & \text{otherwise,} \end{cases}$$

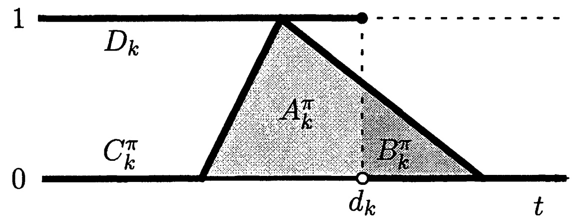

Figure 10.3.

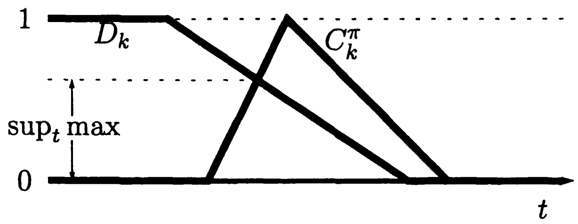

Figure 10.4.

where λ is a given number in the interval $[0,1]$; see Figure 10.4. Again the problem of minimizing the number of λ-tardy jobs is studied, where now job J_k is called λ-tardy in the permutation schedule π whenever

$$\sup_t \min \{C_k^\pi(t), D_k(t)\} \leq \lambda.$$

This problem can be solved by an extension of Moore's algorithm if completion times are triangular fuzzy numbers.

In both cases, the minimum number of λ-tardy jobs and optimal permutations depend on λ. Therefore, strictly speaking, we deal with infinitely many problems. Since there are only finite number of permutations and since the optimal sequencing probably does not react chaotically to changes in the value of parameter λ, it would be of interest to study this dependence and to obtain further algorithmic results.

The above results together with applicability of Moore's algorithm to minimization of the number of tardy jobs in the deterministic case and in the

stochastic case with exponentially distributed processing times suggest the problem of finding general conditions for applicability of Moore's algorithm. In this respect, the following results are established in [121].

Let $\mathcal{P}$ and $\mathcal{D}$ be nonempty sets and let $\overset{\mathcal{P}}{\succ}$ and $\overset{\mathcal{D}}{\succ}$ be asymmetric negatively transitive relations on $\mathcal{P}$ and $\mathcal{D}$, respectively. In other words, $p \overset{\mathcal{P}}{\succ} p'$ implies $p' \overset{\mathcal{P}}{\not\succ} p$, and $p \overset{\mathcal{P}}{\not\succ} p'$ together with $p' \overset{\mathcal{P}}{\not\succ} p''$ imply $p \overset{\mathcal{P}}{\not\succ} p''$, and similarly for $\overset{\mathcal{D}}{\succ}$. We write $p \overset{\mathcal{P}}{\sim} p'$ if both $p \overset{\mathcal{P}}{\not\succ} p'$ and $p' \overset{\mathcal{P}}{\not\succ} p$; if $p \overset{\mathcal{P}}{\succ} p'$ or $p \overset{\mathcal{P}}{\sim} p'$, that is, if $p' \overset{\mathcal{P}}{\not\succ} p$, then we write $p \overset{\mathcal{P}}{\succsim} p'$, and similarly for $\overset{\mathcal{D}}{\succ}$.

Let $t : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ be a binary operation on $\mathcal{P}$ such that

$$\begin{aligned}
&t(p,p') \overset{\mathcal{P}}{\succsim} p,\\
&t(p,p') = t(p',p),\\
&t(p,p') \overset{\mathcal{P}}{\succ} t(p,p'') \ \text{ whenever } \ p' \overset{\mathcal{P}}{\succ} p'',\\
&t(p,t(p',p'')) = t(t(p,p'),p'').
\end{aligned}$$

For every positive integer i, let

$$T_i : \overbrace{\mathcal{P} \times \mathcal{P} \times \cdots \mathcal{P}}^{i \text{ times}} \rightarrow \mathcal{P}$$

be a mapping such that

- $T_1(p) = p$,
- $T_2(p_1,p_2) = t(p_1,p_2)$, and
- $T_i(p_1,p_2,\ldots,p_i) = t(T_{i-1}(p_1,p_2,\ldots,p_{i-1}),p_i)$ for $i \geq 3$.

Notice that

$$T_i(p_{\pi(1)},p_{\pi(2)},\ldots,p_{\pi(i)}) = T_i(p_1,p_2,\ldots,p_i)$$

for every permutation π of $\{1,2,\ldots,i\}$, and

$$T_i(p_1,\ldots,p_{j-1},p_j,p_{j+1},\ldots,p_i) \overset{\mathcal{P}}{\succsim} T_{i-1}(p_1,\ldots,p_{j-1},p_{j+1},\ldots,p_i) \qquad (10.8)$$

for all $1 \leq j \leq i$.

Furthermore, let $\ll$ be a relation between $\mathcal{P}$ and $\mathcal{D}$ such that

- if $p \overset{\mathcal{P}}{\succsim} p'$ and $t(q,p) \ll d$, then $t(q,p') \ll d$, and
- if $d \overset{\mathcal{D}}{\succsim} d'$ and $p \ll d'$, then $p \ll d$.

To simplify the presentation of results on the following generalized version of the problem of minimizing the number of tardy jobs, we use the concept of

a sequence of a finite set defined as follows. A sequence of a nonempty set Q is a one-to-one mapping of $\{1, 2, \ldots, \mathrm{Card}(Q)\}$ onto Q. Now we are ready to formulate an abstract version of problem of minimizing the number of tardy jobs.

Let $\mathcal{J} = \{J_1, J_2, \ldots, J_n\}$ be a set of jobs such that $J_i = (i, p_i, d_i)$ with $i \in \mathcal{N}$, $p_i \in \mathcal{P}$, and $d_i \in \mathcal{D}$. That is, each job J_i is characterized by an ordered triple (i, p_i, d_i). For a sequence π of $\mathcal{N}$, we denote by $\mathcal{T}(\pi)$ the subset of $\mathcal{N}$ defined as follows: $j \in \mathcal{N}$ belongs to $\mathcal{T}(\pi)$ if and only if, for $i \in \mathcal{N}$ such that $j = \pi(i)$,

$$T_i(p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(i)}) \not\ll d_{\pi(i)}.$$

The jobs J_i with $i \in \mathcal{T}(\pi)$ are called *tardy* in the permutation schedule given by sequence π. The problem is to minimize the number of tardy jobs, that is, to find a sequence π on $\mathcal{N}$ such that

$$\mathrm{Card}(\mathcal{T}(\pi)) \le \mathrm{Card}(\mathcal{T}(\sigma)) \quad \text{for every sequence } \sigma \text{ of } \mathcal{N}.$$

Each such sequence is called *optimal.*

EXAMPLE 10.8 Let both $\mathcal{P}$ and $\mathcal{D}$ be the set of all positive numbers, let $\overset{\mathcal{P}}{\succ}$ and $\overset{\mathcal{D}}{\succ}$ be the natural ordering $>$ of real numbers, and let $t(p, p')$ mean $p + p'$. If $p \ll d$ means $p \le d$, then all assumptions are satisfied, and the problem becomes the deterministic problem of minimizing the number of tardy jobs described in Section 2. □

EXAMPLE 10.9 Let $\mathcal{P}$ be the set of random variables with $p : \mathbf{R} \to \mathbf{R}$ such that the probability of the event $p \le t$ is zero for every nonpositive number t, and let $p \overset{\mathcal{P}}{\succ} p'$ means that $E(p) > E(p')$. Let $\mathcal{D}$ be the set of all nonnegative numbers and $\overset{\mathcal{D}}{\succ}$ be the natural ordering $>$ of real numbers. Furthermore, let $t(p, p')$ means $p \oplus p'$ where $\oplus$ is the standard addition of random variables. If $p \ll d$ means $\mathrm{sign}(E(p - d)) = 0$, then we have the minimization of the number of jobs expected to be tardy. □

EXAMPLE 10.10 Recall the fuzzy problem of minimizing the number of λ-tardy jobs, in which the λ-tardiness is defined by (10.7). This problem can also be described as follows.

Let $\mathcal{P}$ and $\mathcal{D}$ be the set of (L, R)-fuzzy intervals defined as follows. An (L, R)-fuzzy interval p belongs to $\mathcal{P}$ if and only if its membership function μ_p is defined by

$$\mu_p(t) = \begin{cases} \max\{0, L_p(\alpha_p - t)\} & \text{if } t \le \alpha_p, \\ \max\{0, R_p(t - \alpha_p)\} & \text{if } t \ge \alpha_p, \end{cases}$$

where α_p is positive number, and L_p and R_p are decreasing functions satisfying $L_p(0) = R_p(0) = 1$. An (L, R)-fuzzy interval d belongs to $\mathcal{D}$ if and only if its membership function μ_d is defined by

$$\mu_d(t) = \begin{cases} 1 & \text{if } t \leq \beta_d, \\ 0 & \text{if } t > \beta_d, \end{cases}$$

where β_d is a positive number. Let $p \overset{\mathcal{P}}{\succ} p'$ and $d \overset{\mathcal{D}}{\succ} d'$ mean $\alpha_p > \alpha_{p'}$ and $\beta_d > \beta_{d'}$, respectively. Furthermore let $t(p, p')$ mean $p \oplus p'$, where $p \oplus p'$ is an (L, R)-fuzzy interval with membership function

$$\mu_{p\oplus p'}(t) = \begin{cases} \max\{0, L_{p\oplus p'}((\alpha_p + \alpha_{p'}) - t)\} & \text{if } t \leq \alpha_p + \alpha_{p'}, \\ \max\{0, R_{p\oplus p'}(t - (\alpha_p + \alpha_{p'}))\} & \text{if } t \geq \alpha_p + \alpha_{p'}, \end{cases}$$

such that

$$L_{p\oplus p'}(t) = \sup\{L_p(t_1) + L_{p'}(t_2) \mid t = t_1 + t_2\}$$

and

$$R_{p\oplus p'}(t) = \sup\{R_p(t_1) + R_{p'}(t_2) \mid t = t_1 + t_2\}.$$

Let $\lambda \in [0, 1)$, and let $p \ll d$ means $B_p(d) < \lambda A_p$, where A_p is the area below the membership function μ_p of p, and $B_p(d)$ is the area of the part A_p to the right of β_d defining d. Let $\mathcal{J} = \{J_1, J_2, \ldots, J_n\}$ be the set of jobs $J_i = (i, p_i, d_i)$, for $i \in \mathcal{N}$, such that $L_{p_1} = L_{p_2} = \cdots = L_{p_n}$ and $R_{p_1} = R_{p_2} = \cdots = R_{p_n}$. Then, the problem becomes the fuzzy problem of minimizing the number of λ-tardy jobs. □

In the following analysis, we assume that

$$d_n \overset{\mathcal{D}}{\succsim} d_{n-1} \overset{\mathcal{D}}{\succsim} \cdots \overset{\mathcal{D}}{\succsim} d_1.$$

Notice that the asymmetry and negative transitivity of $\overset{\mathcal{D}}{\succ}$ guarantee that this assumption causes no loss of generality.

PROPOSITION 10.11 *(Jackson's Lemma) Let $\pi_{\mathcal{D}}$ be a sequence such that $\pi_{\mathcal{D}}(i) = i$ for every $i \in \mathcal{N}$. A sequence π of $\mathcal{N}$ with $\mathcal{T}(\pi) = \emptyset$ exists if and only if $\mathcal{T}(\pi_D) = \emptyset$.*

PROOF. Obviously, it suffices to show that if $\mathcal{T}(\pi) = \emptyset$ for some sequence π, then $\mathcal{T}(\pi_D) = \emptyset$. Suppose that π is such that $\mathcal{T}(\pi) = \emptyset$. Then

$$T_i(p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(i)}) \ll d_{\pi(i)} \quad \text{for every } 1 \leq i \leq n.$$

If $\pi = \pi_D$, then we are done. Let $\pi \neq \pi_D$. Then there is a number $j \in \mathcal{N}$ such that

$$d_{\pi(j)} \overset{\mathcal{D}}{\succ} d_{\pi(j+1)}.$$

Let σ is a sequence obtained from π by the interchange

$$\sigma(i) = \begin{cases} \pi(j+1) & \text{if } i = j, \\ \pi(j) & \text{if } i = j+1, \\ \pi(i) & \text{otherwise.} \end{cases}$$

For $i \notin \{j, j+1\}$, we have

$$T_i(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(i)}) = T_i(p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(i)}) \ll d_{\pi(i)} = d_{\sigma(i)}.$$

For $i = j$, we have $T_i(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(i)}) \ll d_{\sigma(i)}$, because

$$\begin{aligned} T_{i+1}(p_{\pi(1)}, \ldots, p_{\pi(i)}, p_{\pi(i+1)}) &\overset{\mathcal{P}}{\gtrsim} T_i(p_{\pi(1)}, \ldots, p_{\pi(i-1)}, p_{\pi(i+1)}) \\ &= T_i(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(i)}) \end{aligned}$$

and

$$T_{i+1}(p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(i-1)}, p_{\pi(i)}, p_{\pi(i+1)}) \ll d_{\pi(i+1)} = d_{\sigma(i)}.$$

For $i = j+1$, we have $T_i(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(i)}) \ll d_{\sigma(i)}$, because

$$d_{\sigma(i)} = d_{\pi(i-1)} \overset{\mathcal{D}}{\gtrsim} d_{\pi(i)}$$

and

$$T_i(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(i)}) = T_i(p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(i)}) \ll d_{\pi(i)}.$$

Therefore $\mathcal{T}(\sigma) = \emptyset$.

After $O(n^2)$ times of such interchanges, a sequence σ is obtained such that $\mathcal{T}(\sigma) = \emptyset$ and $\sigma = \pi_D$. ∎

Now suppose $\mathcal{T}(\pi_D) \neq \emptyset$. Let k be the smallest number in $\mathcal{N}$ such that $k \in \mathcal{T}(\pi_D)$, and let j be such that $j \in \{1, 2, \ldots, k\}$ and $p_j \overset{\mathcal{P}}{\gtrsim} p_i$ for each $i \in \{1, 2, \ldots, k\}$.

PROPOSITION 10.12 *Let σ be a sequence of $\mathcal{N}$ such that for some $1 \leq l \leq n$,*

- $\{\sigma(1), \sigma(2), \ldots, \sigma(l)\} \subseteq \{1, 2, \ldots, k\} \setminus \{j\}$, *and*
- $d_{\sigma(l)} \overset{\mathcal{D}}{\gtrsim} d_{\sigma(l-1)} \overset{\mathcal{D}}{\gtrsim} \cdots \overset{\mathcal{D}}{\gtrsim} d_{\sigma(1)}$.

Then, $\{\sigma(1), \sigma(2), \ldots, \sigma(l)\} \cap \mathcal{T}(\sigma) = \emptyset$.

PROOF. It is obvious that if $i \in \{\sigma(1), \sigma(2), \ldots, \sigma(l)\} \setminus \{k\}$, then $i \notin \mathcal{T}(\pi_D)$. From (10.8), we have $i \notin \mathcal{T}(\sigma)$ for each $i \in \{\sigma(1), \sigma(2), \ldots, \sigma(l)\} \setminus \{k\}$. Suppose $k \in \{\sigma(1), \sigma(2), \ldots, \sigma(l)\}$. From $p_j \overset{\mathcal{P}}{\gtrsim} p_k$ we have

$$T_{k-1}(p_1, p_2, \ldots, p_{k-1}) \overset{\mathcal{P}}{\gtrsim} T_l(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(l)}).$$

Moreover, since $d_k \overset{\mathcal{D}}{\gtrsim} d_{k-1}$ and $T_{k-1}(p_1, p_2, \ldots, p_{k-1}) \ll d_{k-1}$, we have

$$T_l(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(l)}) \ll d_k,$$

and therefore $k \notin \mathcal{T}(\sigma)$. ■

PROPOSITION 10.13 *There exists an optimal sequence π of $\mathcal{N}$ such that $j \in \mathcal{T}(\pi)$.*

PROOF. Notice that, from Proposition 10.11, $\mathcal{T}(\pi_D) \neq \emptyset$ implies $\mathcal{T}(\pi) \neq \emptyset$ for all optimal sequences π of $\mathcal{N}$. Let σ be an optimal sequence. If $j \in \mathcal{T}(\sigma)$ we are done. Suppose $j \notin \mathcal{T}(\sigma)$. Without loss of generality, we assume that there exists a number l such that

$$d_{\sigma(l)} \overset{\mathcal{D}}{\gtrsim} d_{\sigma(l-1)} \overset{\mathcal{D}}{\gtrsim} \cdots \overset{\mathcal{D}}{\gtrsim} d_{\sigma(1)},$$

and

$$\mathcal{T}(\sigma) = \{\sigma(l+1), \sigma(l+2), \ldots, \sigma(n)\}.$$

Notice that there exists a number $m \in \{1, 2, \ldots, k\}$ such that $m \in \mathcal{T}(\sigma)$.

Now consider a sequence π satisfying

$$\{\pi(1), \pi(2), \ldots, \pi(l)\} = (\{\sigma(1), \sigma(2), \ldots, \sigma(l)\} \setminus \{j\}) \cup \{m\},$$

and

$$d_{\pi(l)} \overset{\mathcal{D}}{\gtrsim} d_{\pi(l-1)} \overset{\mathcal{D}}{\gtrsim} \cdots \overset{\mathcal{D}}{\gtrsim} d_{\pi(1)}.$$

Notice that

$$\{\pi(1), \pi(2), \ldots, \pi(l)\} \cap (\{1, 2, \ldots, k\} \setminus \{j\}) = \{\pi(1), \pi(2), \ldots, \pi(s)\}$$

for some $1 \leq s \leq l$, and thus we have $m \in \{\pi(1), \pi(2), \ldots, \pi(s)\}$ and $j \in \{\sigma(1), \sigma(2), \ldots, \sigma(s)\}$. From Proposition 10.12, we have

$$\{\pi(1), \pi(2), \ldots, \pi(s)\} \cap \mathcal{T}(\pi) = \emptyset.$$

For $s + 1 \leq i \leq l$, we have $\pi(i) = \sigma(i)$. From $p_j \overset{\mathcal{P}}{\gtrsim} p_m$ and

$$\{\pi(1), \pi(2), \ldots, \pi(i)\} = (\{\sigma(1), \sigma(2), \ldots, \sigma(i)\} \setminus \{j\}) \cup \{m\},$$

we obtain

$$T_i(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(i)}) \overset{\mathcal{P}}{\gtrsim} T_i(p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(i)})$$

and

$$T_i(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(i)}) \ll d_{\sigma(i)} = d_{\pi(i)}.$$

Hence, $\text{Card}(\mathcal{T}(\pi)) \leq \text{Card}(\mathcal{T}(\sigma))$, and therefore, π is an optimal sequence of $\mathcal{N}$ such that $j \in \mathcal{T}(\pi)$. ■

Proposition 10.13 guarantees the correctness of the following generalization of Moore's algorithm.

ALGORITHM

STEP 1: Let $Q := N$.

STEP 2: Let $q := \text{Card}(Q)$. Find a sequence π of Q such that

$$d_{\pi(q)} \overset{\mathcal{D}}{\gtrsim} d_{\pi(q-1)} \overset{\mathcal{D}}{\gtrsim} \cdots \overset{\mathcal{D}}{\gtrsim} d_{\pi(1)}.$$

STEP 3: Find the smallest index $1 \leq k \leq q$ such that $\pi(k) \in \mathcal{T}(\pi)$. If no such k exists, go to Step 5.

STEP 4: Find a number $1 \leq j \leq k$ such that $p_{\pi(j)} \overset{\mathcal{P}}{\gtrsim} p_{\pi(i)}$ for each $1 \leq i \leq k$. Let $Q := Q \setminus \{\pi(j)\}$. Return to Step 2.

STEP 5: Return an arbitrary sequence σ of $\mathcal{N}$ such that $\sigma(i) = \pi(i)$ for each $1 \leq i \leq q$.

4.3. Fuzzy Precedence

The problems involving fuzziness in precedence relations have not attracted much interest yet despite a well developed theory of fuzzy relations and a plenty of results on deterministic problems in the presence of precedence relations. Attempts in this directions can be found in [51] and [69]. Here we present recent result of [123] concerning a fuzzy variant of the deterministic problem studied in [134].

There are $n + 1$ jobs $J_*, J_1, J_2, \ldots, J_n$ to be scheduled on a single machine which is continuously available from time $t = 0$ and can process at most one job at at time. The jobs are also available from $t = 0$ and require uninterrupted processing times $p_*, p_1, p_2, \ldots, p_n$. In addition to processing time p_i, two numbers l_i and u_i are associated with each J_i from the set $\mathcal{J} = \{J_1, J_2, \ldots, J_n\}$. These numbers are assumed to satisfy the inequalities $0 \leq l_i \leq u_i$, and they serve to define a fuzzy number whose membership function $D_i : \mathbf{R} \to [0, 1]$ is defined as follows: If $l_i < u_i$, then

$$D_i(x) = \begin{cases} 0 & \text{if } x < l_i, \\ \dfrac{x - l_i}{u_i - l_i} & \text{if } l_i \leq x \leq u_i, \\ 1 & \text{if } u_i < x, \end{cases} \tag{10.9}$$

and if $l_i = u_i$, then

$$D_i(x) = \begin{cases} 0 & \text{if } x < l_i, \\ 1 & \text{if } x \geq u_i. \end{cases} \tag{10.10}$$

Furthermore, we assume that the feasibility of schedules is restricted by the following requirements:

- Jobs $J_1, J_2, \ldots, J_n$ must be processed in the interval $[0, T]$ where

$$T = \sum_{i=1}^{n} p_i.$$

- Job J_* must be processed last.
- For each $J_i \in \mathcal{J}$, the time between the completion time of J_i and the starting time of J_* must be at least l_i time units.

It follows that each feasible schedule can be represented by an ordered pair (π, B) where π is a permutation of the set $\mathcal{N}$ representing the processing order of jobs from $\mathcal{J}$ and B is the starting time of processing J_* such that

$$B \geq \max\left\{ \left(l_{\pi(i)} + \sum_{j=1}^{i} p_{\pi(j)} \right) \middle| \, i \in \mathcal{N} \right\}. \tag{10.11}$$

To evaluate feasible schedules we use three performance measures:

- the makespan,
- the minimum degree of satisfaction with the given fuzzy precedence constraints, and
- the minimum degree of satisfaction with time delays.

The values of these performance measures are denoted by $v_1(\pi, B)$, $v_2(\pi, B)$, and $v_3(\pi, B)$, respectively, and they are defined as follows.

The makespan $v_1(\pi, B)$ of schedule (π, B) is calculated by

$$v_1(\pi, B) = \max\left\{ \max\left\{ \sum_{j=1}^{i} p_{\pi(j)} \middle| \, i \in \mathcal{N} \right\}, B + p_* \right\}.$$

From (10.11), we have

$$v_1(\pi, B) = B + p_*.$$

To simplify the notation, we assume without loss of generality that $p_* = 0$.

To calculate the minimum degree of satisfaction with fuzzy precedence constraints, we assume that the following types of fuzzy precedences are specified; see [51], [69], [124] for more details. For each ordered pair (J_i, J_j) of different jobs from $\mathcal{J}$, a positive number $\mu(i, j)$ less than or equal to 1 is given such

that if $\mu(i,j) < 1$, then $\mu(j,i) = 1$. The value $\mu(i,j)$ expresses the degree of satisfaction or desirability of processing job J_i before job J_j; the higher number represents higher degree of satisfaction. If $\mu(i,j) = \mu(j,i) = 1$, then the jobs are independent in the sense that no preference between the two possible orders exists.

Using the family $\{\mu(i,j)\}$, we define the minimum degree of satisfaction with fuzzy precedence constraints by

$$v_2(\pi, B) = \min\{\mu(\pi(i), \pi(j)) \mid i, j \in \mathcal{N},\ i < j\}.$$

Note that this performance measure does not depend on B.

As already mentioned, the degree of satisfaction with time delay between J_i and J_* is given by D_i, and the minimum degree of satisfaction with time delays is given by

$$v_3(\pi, B) = \min\{D_{\pi(i)}(d_{\pi(i)}(B, \pi)) \mid i \in \mathcal{N}\},$$

where $d_{\pi(i)}(B, \pi) = B - \sum_{j=1}^{i} p_{\pi(j)}$.

We say that a schedule (π, B) *dominates* a schedule (π', B') if

$$v_1(\pi, B) \le v_1(\pi', B'), \quad v_2(\pi, B) \ge v_2(\pi', B'), \quad v_3(\pi, B) \ge v_3(\pi', B'),$$

and at least one of these inequalities is strict; and we say (π, B) and (π', B') are *equivalent* if

$$v_1(\pi, B) = v_1(\pi', B'), \quad v_2(\pi, B) = v_2(\pi', B'), \quad v_3(\pi, B) = v_3(\pi', B').$$

The problem we are interested in is to find a suitable description of the set of nondominated feasible schedules. We solve this problem by proposing a polynomial time algorithm for finding a set of nondominated schedules that represent the whole set (usually uncountable) of nondominated feasible schedules.

First we introduce some more notation. For each function D_i defined by (10.9) or (10.10), we define its inverse $D_i^{-1} : [0,1] \to [l_i, u_i]$ by

$$D_i^{-1}(\alpha) = \alpha u_i + (1 - \alpha) l_i.$$

Suppose the processing order of jobs from $\mathcal{J}$ and the minimum degree of satisfaction with time delays are specified by a permutation π of $\mathcal{N}$ and $\alpha \in [0,1]$. Then, the schedule with the minimum makespan can be represented by $(\pi, B(\pi, \alpha))$, where

$$B(\pi, \alpha) = \max\left\{ D_i^{-1}(\alpha) + \sum_{j=1}^{i} p_{\pi(j)} \;\middle|\; i \in \mathcal{N} \right\}.$$

It follows from the linearity of D_i^{-1} that $B(\pi, \cdot)$ is a piecewise linear function. Moreover, for each permutation π of $\mathcal{N}$,

$$B(\pi, \alpha) < B(\pi, \alpha')$$

whenever $0 \leq \alpha < \alpha' \leq 1$, because each D_i^{-1} is strictly increasing.

Now the main idea and stages of the algorithm can be described as follows:

STAGE 1. Initialization:

- For $i, j \in \mathcal{N}$ with $i \neq j$, let

$$\alpha_{ij} = \frac{(l_i - l_j)}{(u_j - l_j) - (u_i - l_i)}.$$

  Arrange all α_{ij} which are from the open interval $(0, 1)$ in nondecreasing order and rename the resulting q different values so that

$$0 < \beta(1) < \beta(2) < \cdots < \beta(q) < 1.$$

  Let $\beta(0) = 0$ and $\beta(q + 1) = 1$.

- Arrange all $\mu(i, j)$ from the open interval $(0, 1)$ in nonincreasing order and rename the resulting k different values so that

$$0 < \nu(1) < \nu(2) < \cdots < \nu(k) < 1.$$

  Let $\nu(k + 1) = 1$.

- For $1 \leq s \leq k + 1$, construct a precedence graph $G_s = (\mathcal{J}, A_s)$ such that $(J_i, J_j) \in A_s$ if $\mu(i, j) = 1$ and $\mu(j, i) < \nu(s)$.

Now consider the problem of finding a permutation π which minimizes the makespan under the following conditions:

- The minimum degree of satisfaction with the fuzzy precedence constraints is at least $\nu(s)$.
- The minimum degree of satisfaction with time delays is exactly $\alpha \in [0, 1]$.

This problem can be solved by applying Lawler's algorithm to the problem with cost function $f_i(\cdot, \alpha)$ for each job J_i defined by

$$f_i(x, \alpha) = x + D_i^{-1}(\alpha), \quad i \in \mathcal{N},$$

and the precedence graph G_s. Lawler's algorithm returns a permutation π such that

$$\max\left\{ f_{\pi(i)}\left(\sum_{j=1}^{i} p_{\pi(j)}, \alpha\right) \middle| \, i \in \mathcal{N} \right\} = B(\pi, \alpha),$$

is minimized, and the schedule $(\pi, B(\pi,\alpha))$ satisfies $v_2(\pi, B(\pi,\alpha)) \geq \nu(s)$ and $v_2(\pi, B(\pi,\alpha)) = \alpha$.

Notice that, for each $1 \leq t \leq q+1$, there exists a permutation σ_t of $\mathcal{N}$ such that

$$f_{\sigma_t(1)}(x,\alpha) \leq f_{\sigma_t(2)}(x,\alpha) \leq \cdots \leq f_{\sigma_t(n)}(x,\alpha).$$

for all $\alpha \in [\beta(t-1), \beta(t)]$, since $D_i^{-1}(\alpha) \neq D_j^{-1}(\alpha)$ for any $i \neq j$ and for any $\alpha \in (\beta(t-1), \beta(t))$. Essentially, the permutation obtained from Lawler's algorithm depends only on σ_t and the precedence graph G_s, and thus, Lawler's algorithm returns the same permutation for all $\alpha \in [\beta(t-1), \beta(t)]$. The time complexity of Lawler's algorithm is $O(n^2)$.

STAGE 2. For $1 \leq s \leq k+1$ and $0 \leq t \leq q+1$, run Lawler's algorithm with cost function $f_i(\cdot, \beta(t))$ and precedence graph G_s. Denote the resulting permutation by $\pi_{(s,t)}$.

Now at least one schedule from each equivalence class of nondominated feasible schedules can be represented by $(\pi_{(s,t)}, B(\pi_{(s,t)},\alpha))$ for some $\alpha \in [\beta(t-1), \beta(t)]$. The next step is to find an efficient method to exclude schedules which can be dominated, and to do it in such a manner that exactly one schedule of each equivalence class remains undeleted.

To see how this can be realized efficiently, observe the following facts. For $1 \leq s \leq k+1$, any two schedules from

$$\bigcup_{1 \leq t \leq q+1} \{(\pi_{(s,t)}, B(\pi_{(s,t)},\alpha)) \mid \alpha \in [\beta(t-1), \beta(t))\}$$

do not dominate each other, since for $1 \leq t < t' \leq q+1$ and $\alpha \in [\beta(t-1), \beta(t))$ and $\alpha' \in [\beta(t'-1), \beta(t'))$, we have $\alpha < \alpha'$ and

$$B(\pi_{(s,t')}, \alpha') > B(\pi_{(s,t')}, \alpha) \geq B(\pi_{(s,t)}, \alpha).$$

Also notice that

$$B(\pi_{(s,t)}, \alpha) \leq B(\pi_{(s',t)}, \alpha)$$

and

$$v_2(\pi_{(s,t)}, B(\pi_{(s,t)},\alpha)) \leq v_2(\pi_{(s',t)}, B(\pi_{(s',t)},\alpha)),$$

whenever $1 \leq s < s' \leq k+1$, because G_s is a subgraph of $G_{s'}$. Therefore, $(\pi_{(s,t)}, B(\pi_{(s,t)},\alpha))$ with $\alpha \in [\beta(t-1), \beta(t)]$ is dominated by or equivalent to $(\pi_{(s',t')}, B(\pi_{(s',t')},\alpha'))$ with $\alpha' \in [\beta(t'-1), \beta(t')]$ if $s \leq s'$ and $\alpha = \alpha'$ (i.e., $t = t'$). Therefore by finding all $\alpha \in [\beta(t'-1), \beta(t')]$ such that $B(\pi_{(s,t)}, \alpha)) = B(\pi_{(s',t)}, \alpha))$, we can exclude schedule $(\pi_{(s,t)}, B(\pi_{(s,t)}, \alpha)))$ which is dominated by or equivalent to some schedule $(\pi_{(s',t)}, B(\pi_{(s',t)}, \alpha))$. It can be done in $O(n^2)$ for each pair of s and s' with $s < s'$.

STAGE 3. For $1 \leq t \leq q+1$ and for $1 \leq s < s' \leq k+1$, exclude schedule $(\pi_{(s,t)}, B(\pi_{(s,t)}, \alpha))$ with $\alpha \in [\beta(t-1), \beta(t)]$ which is dominated by schedule $(\pi_{(s',t)}, B(\pi_{(s',t)}, \alpha))$.

Therefore, the problem of finding a set of nondominated feasible schedules which contains exactly one schedule from each equivalent class of nondominated feasible schedule can be solved in polynomial time. A straightforward implementation leads to an $O(n^8)$-time algorithm. However, this complexity can obviously be improved.

4.4. Concluding Remarks

Let us return to the general situation and assume that a real number $v_k(C_k^\pi, D_k)$ is assigned to each pair (C_k^π, D_k). As a result, an ordered n-tuple

$$v(\pi) = (v_1(C_1^\pi, D_1), v_2(C_2^\pi, D_2), \ldots, v_n(C_n^\pi, D_n))$$

is associated with each permutation π. The problem of optimal sequencing can now be considered either as a problem with multiple objectives $v_1, v_2, \ldots, v_n$, or it can be transformed to a deterministic problem with scalar-valued objective by using a suitable aggregating mapping. The latter leads to an objective function

$$F(\pi) = A(v(\pi))$$

where A denotes the corresponding aggregating mapping. As a rule, this approach is used either with the operator "min", or with the operator "sum". Studies dealing with other types of aggregation would be of interest both to theorists and practitioners.

Another direction of research arises from the following approach. Usually the pairs (C_k^π, D_k) are first valuated and then an aggregating mapping operator is applied to obtain a crisp objective function. In other words, first fuzziness is dissolved and then crisp data are processed.

It may be of interest to consider the following alternative procedure. Given a permutation π of $\mathcal{N}$, first the corresponding n-tuple

$$((C_1^\pi, D_1), (C_2^\pi, D_2), \ldots, (C_n^\pi, D_n))$$

of pairs of fuzzy intervals is aggregated into a fuzzy set $A'(\pi)$, and then a real number $F'(\pi) = v'(A'(\pi))$ is assigned to $A'(\pi)$. Numbers $F(\pi)$ and $F'(\pi)$ may differ but both functions induce an ordering on the set of permutation schedules.

Following Fodor and Roubens [32], we can form a valued binary relation on the set of permutation schedules equipped with a relation "better than" as follows. The set of permutation schedules is viewed as "finite set of alternatives", and the set of jobs is viewed as the set of "individuals with preferences". For

each job J_k, a valued binary relation B_k on the set of permutation schedules is formed by requiring

- $B_k(\pi, \sigma) = 0$ if π and σ are *equally good*,
- $B_k(\pi, \sigma) = 1$ if π is *definitely better than* σ,
- $B_k(\pi, \sigma) > B_k(\pi', \sigma')$ if π is better than σ is *more credible than* π' is better than σ'.

Obviously, if $B_k(\pi, \sigma) \in \{0, 1\}$ for all (π, σ), then B_k is a crisp relation. Since permutation schedules π and σ determine ordered n-tuples

$$(C_1^\pi, C_2^\pi, \ldots, C_n^\pi), \quad (C_1^\sigma, C_2^\sigma, \ldots, C_n^\sigma)$$

of completion times, we can define relations $B_1, B_2, \ldots, B_n$ by

$$B_k(\pi, \sigma) = \sup_{x \geq y} \min\{C_k^\pi(x), C_k^\sigma(y)\}.$$

In the aggregation phase we can use either the operator

$$B = \min\{\varphi_1(B_1), \varphi_2(B_2), \ldots, \varphi_n(B_n)\}$$

where functions $\varphi_k : [0,1] \to [0,1]$ are nondecreasing and such that

- $\varphi_k(1) = 1$ for each k,
- $\varphi_k(0) = 0$ for at least one k;

or the operator

$$B = \max\{\varphi_1(B_1), \varphi_2(B_2), \ldots, \varphi_n(B_n)\}$$

where functions $\varphi_k : [0,1] \to [0,1]$ are nondecreasing and such that

- $\varphi_k(1) = 1$ for at least one k,
- $\varphi_k(0) = 0$ for each k.

The resulting relation then becomes the starting point of a defuzzification procedure.

Probably most remarkable is the fact that almost all papers deal with permutation schedules only. It is remarkable because it may be advantageous to allow for the inserted machine idle time whenever the membership functions of fuzzy due dates are not monotone. It is also remarkable because the permutation schedule is a crisp concept. It is certainly desirable to study also fuzzy solutions to fuzzy scheduling problems. Another underdeveloped area is the investigation of dynamic schedules in fuzzy environment.

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